# ON PROPERTIES OF A DOMAIN FOR WHICH ANY DERIVED SET IS CLOSED*

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**1. Introduction.** FRÉCHET, in his thesis†, defines a general class of assemblages $(L)$ of objects of any sort, with which some definition of limiting element of a sequence is associated.‡

The most interesting type of such assemblages is that which FRÉCHET calls *compact.*§ Among such assemblages one very important class are those which have the property that the *first derived set* $(E')$ *of every subset* $(E)$ *of* $(D)$ *is closed.*‖

In this paper, some of the properties of any fundamental assemblage $(D)$ which has both the properties described above will be discussed. We shall call $(D)$ the fundamental domain.¶

**2. Elements Interior to Assemblages.** Let there be given a fundamental domain $(D)$ with the properties** described in § 1. An element $A$ is said to be *interior*†† to a set $(E)$ of $(D)$ if $A$ is not a limiting element of any sequence of elements of $(D)$ which do not belong to $(E)$. One set $(E)$ is interior to another set $(F)$ if every element of $(E)$ is interior to $(F)$.

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† Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906).

‡ The only restrictions are (1) that a sequence of identical elements, $A, A, \cdots, A, \cdots$ be said to have $A$ as a limiting element, (2) that if a sequence $A_1, A_2, \cdots, A_n, \cdots$ has a unique limit, any infinite subsequence selected from it, in the same order, has the same limit. (FRÉCHET, loc. cit., § 7.)

§ A fundamental domain is *compact* if every infinite set of its elements has at least one limiting element. (FRÉCHET, loc. cit., § 9.)

‖ Compare FRÉCHET, loc. cit., § 22. It is evident that domains exist which do not have this property. FRÉCHET cites the case of the domain of all continuous functions defined in a closed interval. See also SCHOENFLIES, *Punktmannigfaltigkeiten*, Bericht der Deutschen Mathematiker-Vereinigung, Ergänzungsband II (1908), page 86, lines 1–6, and page 279. On page 282 it is proved that this property is not a consequence of the compactness of $(D)$.

¶ The notation $(L)$ is used by FRÉCHET for any domain for which an acceptable definition of limit is given. In this paper we shall assume that $(D)$ is *compact* and has the closure property mentioned above. The investigation of the properties of sets having these two characteristics is suggested by SCHOENFLIES, loc. cit., p. 279.

** The following theorems involve this hypothesis, which is not repeated below.

†† See FRÉCHET, loc. cit., § 8.

An immediate consequence of this definition is the following theorem:

THEOREM 1. *If an element $A$ is interior to a set $(E)$, all but a finite number of elements of any sequence whose limiting element is $A$ are interior to $(E)$.* For if not, let $A_1, A_2, \cdots, A_p, \cdots$ be the infinite set of elements not interior to $(E)$ selected from such a sequence. Since $A_i$ is not interior to $(E)$, it can be approached by a sequence of elements $A_i^{(1)}, A_i^{(2)}, A_i^{(3)}, \cdots, A_i^{(k)}, \cdots$ of $(D)$ which do not belong to $(E)$. The derivative of the set of elements $A_i^{(k)}$ contains the elements $A_i$. Since this derivative is closed, $A$ is one of its elements; hence $A$ can be approached by a sequence of elements of the set $A_i^{(k)}$ none of which belong to $(E)$. This is contrary to the hypothesis that $A$ is interior to $(E)$, and the theorem is proved.

**3. The Heine-Borel Theorem.** The theorem just proved enables us to show that the Heine-Borel theorem * holds for a fundamental domain $(D)$ which has the properties of section 1.

THEOREM 2. *Let $(E_1), (E_2), \cdots, (E_n), \cdots$ be a countable set of assemblages, such that every element of a certain closed assemblage $(K)$ is interior to at least one of the assemblages $(E_i)$; then every element of $(K)$ is interior to one of a finite number of the assemblages $(E_i)$.*

For suppose the theorem untrue. Let $A_1$ be an element of $(K)$ not interior to $(E_1)$, and let $(E_{n_1})$ be the first of the $(E_i)$ to which $A_1$ is interior. Let $A_2$ be an element not in $(E_j), j \leqq n_1$; and let $(E_{n_2})$ be the first $(E)$ to which $A_2$ is interior. Continuing this process indefinitely, as we may in case the theorem is untrue, we obtain an infinite sequence of elements $A_i$, such that $(E_{n_i})$ is the first $(E)$ to which $A_i$ is interior, and $n_i > n_{i-1}$. The sequence $A_i$ has a limiting element $A$, which is itself interior to some $(E)$, say $(E_q)$. We may suppose $A$ is the only limiting element of the $A_i$, for, if not, a partial sequence can be selected out of the $A_i$ which has $A$ as its only limiting element and for which the essential properties of the $A_i$ are preserved. Since $A$ is then the only limiting element of the $A_i$, it follows from Theorem 1 that all but a finite number of the $A_i$ are interior to $(E_q)$. This shows that the original process is fallacious, since $n_i = q$ for all sufficiently large values of $i$. Hence the theorem as stated must be correct.

**4. Examples: The Hilbert Space.** As an instance of a fundamental domain to which the preceding theorems apply we may cite, of course, any limited region of an ordinary space of one dimension (or in fact of $n$ dimensions), with the usual definition of limiting point. In fact, any assemblage of points in $n$

* The theorem is restricted to the case in which sets $E_i$ form a *countable* assemblage; this was the original form of the BOREL statement, afterwards generalized, for real variables, by LEBESGUE and others. FRÉCHET gives (loc. cit., §§36, 42) proofs for any domain of type $(V)$ (*ibid.*, § 27, et seq.), under restrictions on the definition of distance (voisinage) and of limiting element which include the conditions of this paper (*ibid.*, § 28). The more general form of the theorem occurs in this paper as part of Theorem 16.

dimensions which is itself closed may be chosen as the fundamental domain $(D)$. Such a domain evidently has the property that the first derived assemblage of any subset $(E)$ is closed. A more characteristic example, in which the definition of distance has nothing to do with the definition of limiting object is the Hilbert space of an infinite number of dimensions;* the objects used are the points $(x_1, x_2, \cdots, x_n, \cdots)$ with the restriction $\Sigma x_i^2 \leqq 1$. The definition of distance between two points is $[\Sigma (x_i^{(1)} - x_i^{(2)})^2]^{\frac{1}{2}}$; and the points $(x^{(1)}), (x^{(2)}), \cdots, (x^{(n)}), \cdots$ approach the point $(x)$ as a limit if $\lim_{n=\infty} x_i^{(n)} = x_i$ for every value of $i$.† This space evidently possesses the properties of § 1; yet the set of points $(x^{(n)})$ which *approach* $(x)$ may all lie *at unit distance* from $(x)$, as in the example $x_i^{(n)} = 0$ if $n \neq i$, $x_n^{(n)} = 1$; $x_i = 0$. FRÉCHET has given a definition of distance (voisinage) for which the preceding definition of limit points makes the distance between $(x^{(n)})$ and $(x)$ approach zero.‡ The preceding work shows that it is not always necessary to set up a definition of distance for the Hilbert space; for other domains of objects to do so might be very difficult or even impossible.

**5. Other Theorems.** Another property of sets of real numbers which holds for a fundamental domain $(D)$ having the properties of section 1 is the following:§

THEOREM 3. *The set of elements at which the oscillation of any limited function $f(x)$ is not less than a given number $k$ is closed.* Let $S$ be any sequence of elements $A_1, A_2, A_3, \cdots, A_i, \cdots$ whose only limiting element is $A$; let $M_s(A, n)$ be the upper limit of the values of $f(x)$ at the points $A_{n+p} (p = 1, 2, \cdots)$; then $M_s(A) = \lim_{n=\infty} M_s(A, n)$ surely exists and is called the *maximum* of $f(x)$ at $A$ with respect to $S$. The *maximum* of $f(x)$ at an element $A$ is defined to be the upper limit of the maxima at $A$ with respect to all possible sequences of elements different from $A$ whose only limiting element is $A$. The *minimum* at $A$ is similarly defined. The *oscillation* at $A$ is the difference between the maximum and the minimum.‖

Let $L$ be a limiting element of a sequence $A_1, A_2, \cdots, A_n, \cdots$ such that the oscillation of $f(x)$ at $A_n$, for all values of $n$, is not less than $k$. If the maxima $M_n$ of $f(x)$ at $A_n$ are not less than a fixed number $H$ for an infinite

* HILBERT, *Wesen and Ziele einer Analysis der unendlichvielen unabhängigen Variabeln*, Rendiconti del Circolo Matematico di Palermo, vol. 27 (1909). Also see HILBERT's *Mitteilungen über die Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (vierte und fünfte) Göttinger Nachrichten, 1906.

† HILBERT, Palermo Rendiconti, loc. cit., p. 3.

‡ FRÉCHET, Nouvelles Annales de Mathématiques, 4th series, vol. 8, March and July, 1908. Strictly, by the wording of pages 1 and 23 (July), it would appear that the Heine-Borel theorem does not hold for the Hilbert space; this is however only due to a change of nomenclature, as a closer inspection enables the reader to prove.

§ Compare, for example, BAIRE, *Leçons sur les fonctions discontinues*, p. 73.

‖ Compare BAIRE, loc. cit., p. 70; and a paper by the writer, Bulletin of the American Mathematical Society, vol. 13 (1907), p. 378.

number of values of $n$, it follows that the maximum $M_L$ of $f(x)$ at $L$ is not less than $H$; for if $n_1, n_2, \cdots, n_i, \cdots$ are the values of $n$ for which $M_{n_i} \geqq H$, we can select a set of elements $A_{n_i}^{(1)}, A_{n_i}^{(2)}, \cdots, A_{n_i}^{(j)}, \cdots$ which approach $A_{n_i}$, and for which $f(x)$ exceeds $H-\epsilon$, where $\epsilon$ is an arbitrarily small fixed number independent of $i$. Since $L$ is, by section 1, in the first derivative of $A_{n_i}^{(j)}$, the maximum at $L$ is not less than $H$.

Likewise if the minima $m_n$ of $f(x)$ at $A_n$ are not greater than $K$ for an infinite number of values of $n$; the minimum $m_L$ of $f(x)$ at $L$ is not greater than $K$. Suppose that the maximum $M_L$ and the minimum $m_L$ of $f(x)$ at $L$ differ by $k-4\delta$. Then there are surely only a finite number of values of $n$ for which $M_n$ exceeds $M_L+\delta$ and only a finite number of values of $n$ for which $m_n$ is less than $m_L-\delta$; hence there are only a finite number of values of $n$ for which $M_n-m_n$ exceeds $k-2\delta$, which is contrary to hypothesis. Therefore the oscillation, $M_L-m_L$, of $f(x)$ at $L$ is not less than $k$; and the theorem is proved. It results also from the proof just given that *:

THEOREM 4. *The maximum of $f(x)$ is itself a function which is semi-continuous above; and the minimum is semi-continuous below.*

Another theorem which remains true is:

THEOREM 5. *The frontier of any set is closed.*

The *frontier* of a set $(E)$ of $(D)$ is defined to be the elements of $(E)$ which are limiting elements of elements not in $(E)$ together with the elements not in $(E)$ which are limiting elements of elements of $(E)$. Let $A$ be any limiting element of elements of the frontier. If $A$ is a limiting element of frontier elements $f_1, f_2, \cdots, f_n, \cdots$ which belong to $(E)$, each $f_i$ is a limiting element of elements not in $E$; hence $A$ is a limiting element of elements not in $E$. Hence $A$ is a limiting element of elements of $(E)$ and also of elements not in $(E)$; it is therefore a frontier element. A similar argument applies in case $A$ is a limiting element of frontier elements not in $(E)$.

That the several theorems proved above hold true without any restrictions on the definition of distance, or any connection between that definition and the definition of limiting element, is especially remarkable because the proofs of the analogous theorems for real numbers apparently make essential use of the distance notion. That such is the case, however, is demonstrated here by the entire absence of any definition of distance.†

It is also clear that the property emphasized in section 1 — that the derivative of any set be closed — may be used to characterize a type of domains (with assigned definitions of limiting elements) which have in common some fundamental properties.

* Compare BAIRE, loc. cit., p. 72.

† See SCHOENFLIES, *Bericht*, loc. cit., p. 289. Compare, however, VEBLEN, these Transactions, vol. 6 (1905), p. 167.

Another important theorem which follows from these same assumptions occurs incidentally in the next section.

6. **Enclosable Domains.** In the theory of linear point sets the following statement may be taken as an axiom: If the interval $(A_i, B_i)$ is interior to the interval $(A_{i-1}, B_{i-1})$ and if the distance $A_i B_i$ approaches zero, there is a single point common to all these intervals.

We shall now assume the analogous property: *Corresponding to any element $A$ of $(D)$ there exists a countable set of closed assemblages $Q_i(A)$ which contain the single common interior element $A$, and which have the following properties:*

(*a*) *$Q_i(A)$ is interior to $Q_{i-1}(A)$.*

(*b*) *Corresponding to any integer $m$ an integer $n$ exists, such that, for any point $B$ whatever, any $Q_n$ which contains $B$ is interior to $Q_m(B)$.*

When the domain $D$ with its definition of limiting element satisfies this requirement, we shall say that $D$ has the *enclosable** property. All of the assemblages $Q$ are supposed to be selected now, and are held fixed in all that follows. Any $Q_i$ is said to be of *rank $i$*.

Several consequences of these assumptions may be stated at once:

LEMMA 1. *The fundamental domain $(D)$, if it contains more than a single element, is perfect.* For any isolated point of $(D)$ is "interior" to any assemblage whatever. And $(D)$ is closed, by its definition.

LEMMA 2. *If $A_i$ is interior to $Q_i(B)$, for all values of $i$, $B$ is the only limiting element of the elements $A_i$.* For since $B$ is the only element interior to all the $Q_i(B)$, any element $C$ distinct from $B$ may be enclosed in $Q_j(C)$, so that $Q_j(C)$ and $Q_i(B)$ have no elements in common for sufficiently high values of $i$ and $j$. Then the lemma follows from Theorem 1.

It follows also from Theorem 1, that *a necessary and sufficient condition that $\mathrm{Lim}_{i=\infty} A_i = B$ is that at least one of the $A_i$ lies in $Q_j(B)$ for every value of $j$.*

LEMMA 3. *If a sequence $A_i$ have the single limiting element $B$, corresponding to any fixed integer $m$, an integer $N$ exists such that $B$ is interior to $Q_m(A_{N+p})$ for all values of $p$.*

For a given $m$ select $n$ by $(b)$; since $Q_n(B)$ contains $A_{N+p}$, where $N$ is determined by Theorem 1, $Q_n(B)$, and therefore $B$ itself, is interior to $Q_m(A_{N+p})$, by $(b)$.

LEMMA 4. *If $B$ is interior to $(E)$, a number $m$ exists such that any $Q_m$ which contains $B$ is interior to $(E)$.*

* My attention is called to the similarity between this hypothesis and that given by VEBLEN for point sets, these Transactions, vol. 6 (1905), p. 168. The notion of uniformity of enclosure is essential in both cases. It is suggested that the word "enclosable" might read "uniformly enclosable"; the former term is retained here, however, for simplicity.

For let $R_i$ be a point of $Q_i(B)$ which is not interior to $(E)$. By Lemma 2, $\text{Lim}_{i=\infty} R_i = B$, and all but a finite number of the $R_i$ are interior to $(E)$ by Theorem 1. It follows from $(b)$ that $m$ exists as stated in the theorem.

LEMMA 5. *If $B$ is interior to $(E)$ and if $\text{Lim}_{i=\infty} A_i = B$, where $A_i$ is an element interior to a certain assemblage $Q_i'$ of the type $Q_i$ for every value of $i$, then $n$ exists such that $Q_{n+p}'$ is interior to $(E)$ for all values of $p$.*

For $Q_i(A_j)$ contains $B$ and lies in $(E)$ for $i = m$, $j \geqq N$ by Lemma 4 and Lemma 3. Corresponding to $m$, a number $n$ exists, by $(b)$, such that any $Q_n$ which contains $A_j$ is interior to $Q_m(A_j)$; hence $Q_{n+p}'$ is interior to $Q_m(A_{n+p})$ and therefore to $(E)$, if also $n \geqq N$.

THEOREM 6. *If any sequence $Q_i'$ of assemblages of the types $Q_i$ $(i = 1, 2, 3, \cdots, n, \cdots)$ are such that $Q_i'$ is interior to $Q_{i-1}'$, there is one and only one point $B$ interior to all $Q_i'$, and also $\text{Lim}_{i=\infty} A_i = B$ if $A_i$ is any element interior to $Q_i'$.*

For if $A_i$ is interior to $Q_i'$ the sequence $A_i$ has at least one limiting element, $B$, and $B$ is interior to every $Q_i'$. Any other element $C$ distinct from $B$ can be enclosed by $Q_j(C)$ so that $Q_j(C)$ and $Q_i(B)$ have no elements in common. But $Q_i(B)$ contains $Q_k'$ for *some* value of $k$, by Lemma 5; hence $Q_j(C)$ contains at most a finite number of the points $A_i$. It follows that $B$ is the only element common to all the $Q_i'$, and is the only limiting element of $A_i$.

THEOREM 7. (*Cauchy Fundamental Theorem.*) *The necessary and sufficient condition that the sequence $A_j$ have a single limiting element is that, corresponding to any arbitrarily assigned value of $i$, a number $n$ exists such that, for all values of $p$, all the elements $A_{n+p}$ are interior to a single assemblage of the type $Q_i$.*

The condition is necessary. For if $\text{Lim}_{j=\infty} A_j = B$, a number $m$ exists by Lemma 3, such that $Q_i(A_m)$ contains $B$; hence $Q_i(A_m)$ contains $A_{n+p}$ for sufficiently high values of $n$, by Theorem 1.

The condition is sufficient. For if $Q_i$ contains $A_{n+p}$, the sequence $A_{n+p}$ has at least one limiting element $B$; if there were another such limiting element $C$, $Q_k(C)$ and $Q_k(B)$ would contain no common element for sufficiently high values of $k$; but $Q_k(B)$ contains $Q_i$, by $(b)$, for sufficiently high values of $i$, whence $Q_k(C)$ contains at most a finite number of the elements $A_j$, and is therefore not a limiting element.

THEOREM 8. *The set $P$ of elements of condensation * of any condensed † assemblage $E$ is either vacuous or perfect.*

It can be shown that $P$ is *closed* without the assumption of the enclosable

* A limiting element is an element of condensation if it remains a limiting element after any countable assemblage is removed. See FRÉCHET, loc. cit., p. 6, § 8.

† A condensed set is one for which any non-countable set of elements gives rise to at least one point of condensation. See FRÉCHET, loc. cit., p. 19, footnote.

property. For let $A$ be a limiting element of elements of $P$, then $A$ belongs to $(E')$, since $P$ belongs to $(E')$, and $(E')$ is closed. If a countable set of elements are removed from $(E)$, $P$ remains intact; hence $A$ remains a limit point of $P$, and $A$ remains a limit point of the remaining $(E)$. Hence $A$ belongs to $(P)$.*

Let us now show that $(P)$ is dense on itself; let $A$ be any point of $(P)$. Surround $A$ by a set of assemblages $(Q_i)$. Then there are an infinite number of values of $i$, $i_1, i_2, \cdots, i_n, \cdots$, for which a non-countable set of elements of $(E)$ are interior to $(Q_{i-1})$, but not to $(Q_i)$, since otherwise a value of $i$ exists for which only a countable set of elements of $(E)$ are interior to $Q_i$, and this, by Theorem 1, contradicts the hypothesis that $A$ belongs to $(P)$. Since $(E)$ is condensed, there is therefore at least one element of $(P)$ in $Q_{i_n-1}$, but not in $Q_{i_n+1}$, for all values of $i_n$. Let these be $p_1, p_2, \cdots, p_n, \cdots$. Then $A$ is the limit point of $p_n$, by Theorem 6.

THEOREM 9. *The set of elements $(C)$ of any condensed assemblage $(E)$ which are not elements of condensation is countable.*

For every element $k$ of $(C)$ is such that no element of condensation lies in $Q_i(k)$, where $i$ can be found when $k$ is given. The set of elements $(C_i)$ for which $i$ has the same fixed value is countable, for otherwise an element of condensation $p$ determined by $(C_i)$ would lie in $Q_i(k)$ for some element $k$ of $(C_i)$, by Lemma 3. Hence the entire assemblage $(C)$ is countable.

It follows that *every* CLOSED *condensed assemblage $(E)$ consists of a countable set and a set which is either vacuous or perfect.* The conclusion is still valid if $(E)$ is not closed but contains all its points of condensation.

Finally, any condensed assemblage $(E)$ consists of a countable set and a set of which every point is a point of condensation; the latter set is dense on itself.

With a modification of Fréchet's proof (Thesis, p. 20) similar to the modification of Lindelöf's proof† of Theorem 9 above, it can be shown that

THEOREM 10. *The elements of any assemblage $(E)$ which do not belong to its first derived assemblage $(E')$ form a countable set.*

Hence, if any derived assemblage is countable, the original assemblage is countable.

Precisely following the proofs of Fréchet (Thesis, pages 20–22) we may now show that:

THEOREM 11. *Any closed assemblage $(E)$ is the first derivative of some assemblage selected from $(D)$.*

THEOREM 12. *Any closed sub-assemblage $(F)$ of an assemblage $(E)$ can be formed from $(E)$ by omitting from $(E)$ the points interior to a countable number of the assemblages $Q$.*

*See FRÉCHET, Palermo Rendiconti, vol. 30 (1910), p. 23, footnote. The closure property of section 1 above is the principle needed in FRÉCHET's thesis, p. 19, § 29.

† BOREL, *Leçons*, page 5.

On account of its importance in what follows we shall repeat Fréchet's proof, which requires alteration in certain details, in the following special case:

If we select $(F)$ to be vacuous and $(E) \equiv (D)$, it results that

THEOREM 13. *Every point of $(D)$ is interior to at least one of a countable set of assemblages $Q$.**

Select some $Q$ at random, say $Q^{(0)}$. Then any point $A$ of $(D)$ not belonging to $Q^{(0)}$ has the property that $Q_i(A)$ has no point in common with $Q^{(0)}$ for sufficiently high values of $i$. We shall show that there exists a countable set of elements of $(D)$:

$A_1^{(1)}, \cdots, A_1^{(p_1)}; A_2^{(1)}, \cdots, A_2^{(p_2)}; \cdots A_n^{(1)}, \cdots, A_n^{(p_n)}; \cdots$ such that $(a)$ $Q_n(A_n^{(j)})$ has no point in common with $Q^{(0)}$, and $(b)$ any element of $(D)$ belongs to $Q^{(0)}$ or is interior to $Q_n(A_n^{(j)})$ for some values of $n$ and $j$. For suppose the process carried out to any stage, so that the points up to $A_{n-1}^{(p_{n-1})}$ are selected. Suppose $A$ exists such that $Q_n(A)$ contains no point in common with $Q^{(0)}$ and that $A$ is not interior to $Q_k(A_k^{(j)})$ for $k < n$. Take this $A$ to be $A_n^{(1)}$; and select another $A$ of $(D)$ which has the same properties and which is not interior to $Q_n(A_n^{(1)})$. Continuing in this way, form $A_n^{(1)}, A_n^{(2)}, \cdots$. This suite is finite, for if not the limit point $A_n^{(0)}$ would exist and $Q_n(A_n)$ would contain $A_n^{(0)}$ by Lemma 3, and therefore also $A_n^{(m+p)}$, by Theorem 1, for sufficiently high values of $m$.

This completes the proof if we note that every point of $Q^{(0)}$ is interior to that $Q$ which precedes $Q^{(0)}$ in the sequence of $Q$'s which belong to the same point $A_0$ as does $Q^{(0)}$.

Since every $Q_n(A_n^{(j)})$ may be replaced by $Q_k$, where $k < n$, since $Q_n(A_n^{(j)})$ is interior to $Q_k(A_n^{(j)})$, it follows that every point of $(D)$ is interior to at least one of a countable set of assemblages $Q_k$ *of the same rank, $k$.*

But, in the preceding proof, all of the points up to and including $A_{n-1}^{(p_n-1)}$ may be omitted; hence it follows that

THEOREM 14. *Every element of the fundamental domain $(D)$ is interior to a countable number of the assemblages $Q_k$, for any preassigned value of $k$.*

Hence also, by Theorem 2, every element of $(D)$, or of any closed subset of $D$, is interior to a *finite* number of the assemblages $Q_k$, for any preassigned value of $k$.

THEOREM 15. *The domain $(D)$ itself is the first derived assemblage of a certain countable set of its own elements; that is, $(D)$ is "separable."*†

For let us take the set of elements

$$A_0; A_1^{(1)}, \cdots, A_1^{(p_1)}; A_2^{(1)}, \cdots, A_2^{(p_2)}; \cdots$$

defined in the proof of Theorem 13, and the elements

$$A_{0_k}; A_{(k+1)_k}^{(1)}, \cdots, A_{(k+1)_k}^{(p_{k+1})_k}; A_{(k+2)_k}^{(1)}, \cdots, A_{(k+2)_k}^{(p_{k+2})_k}; \cdots$$

*See FRÉCHET, Thesis, p. 25.

†See FRÉCHET, Thesis, p. 23.

obtained in the same proof by omitting the points $A_1^{(1)}, \cdots, A_k^{(p_k)}$ and by choosing $Q^{(0)}$ to be at least of rank $k+2$, as in the proof of Theorem 13. All of these points form a countable set; we shall show that any element of $(D)$ is a limiting element of this set.

For let $A$ be any element of $(D)$. By Theorem 13, $A$ is interior to $Q_n(A_n^{(j)})$ for definite values of $n$ and $j$. With this fixed value of $n$, let us select $m$ by Lemma 4 so that any $Q_m$ which contains $A$ lies in $Q_n(A_n^{(j)})$. Then choosing $k \geqq m$, select $r$ so that $A$ lies in $Q_{k+r}(A_{(k+r)_k}^{(s)})$. Proceeding in the same way, we may locate $A$ in the interior of a sequence of assemblages $Q^{(1)}$, $Q^{(2)}, \cdots$ such that $Q^{(n)}$ is of higher rank than $Q^{(n-1)}$ and interior to it, and such that $Q^{(n)}$ contains a point $A^{(n)}$ of the countable set selected above. It follows, by Theorem 6, that $\operatorname{Lim}_{n=\infty} A^{(n)} = A$.

COROLLARY. *Any subset of $(D)$ which is dense on itself is contained in the first derived assemblage of a countable set of its own elements.*

THEOREM 16. *Let $(E)$ be an assemblage, each of whose elements is interior to at least one of a family $(H)$ of assemblages $I$, where $(H)$ is not necessarily countable; then the necessary and sufficient condition that every point of $(E)$ is interior to at least one of a finite number of the assemblages $I$ is that $(E)$ be closed.*

The condition is sufficient. For, since every element $e$ of $(E)$ is interior to some assemblage $I$, there is an upper limit to the value of $n$ for which $Q_n(e)$ is not interior to some $I$. If not, let $e_1, e_2, \cdots, e_n, \cdots$ be a set of elements of $(E)$ such that $Q_n(e_n)$ is not interior to any $I$. Let $e_0$ be one of the limiting elements of $e_n$; we may suppose without loss of generality that the $e_n$ have been already selected so that $\lim_{n=\infty} e_n = e_0$. Let $I_0$ be an assemblage of $(H)$ to which $e_0$ is interior. Then $Q_{n+p}(e_{n+p})$ is interior to $I_0$ for sufficiently large values of $n$, by Lemma 5.

Let $N$ be the upper limit on $n$ for which $Q_n(e)$ is not interior to some $I$. Select the finite set of assemblages $Q$ of rank $N+1$ for which every point of $(D)$ is interior to some $Q_{N+1}$, by Theorem 14 and Theorem 2. Let $Q^{(1)}, Q^{(2)}, \cdots, Q^{(k)}$ be the subset of these $Q$'s each of which contain at least one point of $(E)$. Select $I^{(1)}, I^{(2)}, \cdots, I^{(k)}$ such that $Q^{(i)}$ is interior to $I^{(i)}$; then every point of $(E)$ is interior to one of this finite set of $I$'s.*

The condition is also necessary. For if $(E)$ is not closed, let $e_1, e_2, \cdots, e_n, \cdots$ be a set of elements of $e$ whose only limiting element $e_0$ does not belong to $(E)$; and consider the family $(H)$ of assemblages $I_n$ whose elements belong to $(E)$ but not to $Q_n(e_n)$. Not all the points $e_i$, and, *a fortiori*, not all the points of $(E)$ can belong to a finite number of the $I_n$, by Theorem 7.

* This proof follows in spirit that of FRÉCHET, Thesis, p. 26, but in style it follows a proof by BAGNERA, Rendiconti del Circolo Matematico di Palermo, vol. 28 (1909), p. 244, which is essentially the same as FRÉCHET's.

THEOREM 17. *Every non-countable assemblage* $(E)$ *of* $(D)$ *is condensed.*

Given a non-countable set of elements of $(E)$, at least one of the finite set of assemblages $Q_k$ of rank $k$, to at least one of which each point of $(E)$ belongs, contains a non-countable set of elements of $(E)$. Hence a sequence $Q_1^{(1)}$, $Q_2^{(2)}$, $Q_3^{(3)}$, $\cdots$, $Q_n^{(n)}$, of the type $Q$ exists, such that $Q_n^{(n)}$ is interior to $Q_{n-1}^{(n-1)}$ and that $Q_n^{(n)}$ contains a non-countable set of elements of $(E)$. Hence, by Theorem 6, a point $A$ exists which is the only point interior to all of the $Q_n^{(n)}$ and which is a point of condensation of $(E)$.

From Theorems 9 and 17 we now conclude:

THEOREM 20. *Every closed assemblage* $(E)$ *consists of a countable set together with a set which is perfect.*

We have now proved a set of theorems analogous to the principal fundamental theorems of the theory of bounded sets of points; for these the only assumptions made were those of § 1 and § 6. The hypothesis that the first derived assemblage of every set is closed is therefore seen to be a fundamental hypothesis of far-reaching character. The fundamental domains which possess that property include many important examples besides the original bounded sets of points, and have characteristics in common which make that class of domains one of great importance.

It should also be noticed that the enclosable property of section 6 is possessed by all ordinary domains under any of the usual definitions of distance, if $Q_i(A)$ be taken to be the spheroid about $A$ as center, with radius $1/i$.

# IMPORTANT COVARIANT CURVES AND A COMPLETE STSTEM OF INVARIANTS OF THE RATIONAL QUARTIC CURVE*

BY

J. E. ROWE

## *Introduction.*

It is well understood that different domains of rationality are useful in discussing different properties of curves. Two domains are employed in the following, to render possible the geometric interpretation of certain invariants and covariant loci of the rational plane quartic.

Section 1 is divided into two parts: In the first part is given a straightforward proof of the covariance of curves derived from $R^n$ by a certain translation scheme; in the second part SALMON's work on the combinants of two binary quartics is applied to those covariant curves of the $R^4$ which can be found as combinants. In Section 2 the most important invariants of the $R^4$ are discussed, and four invariants are found in terms of which any other invariant relation on the $R^4$ can be expressed algebraically; in this sense these four invariants constitute an *algebraically complete system.* Section 3 contains a treatment of the invariants of the $R^4$ when the $R^4$ is taken as the section of the Steiner Quartic Surface by a plane; in this scheme the invariants occur as symmetric functions of the coefficients of the cutting plane.

### § 1*a*. *Certain Covariants of Rational Curves.*

Let the rational curve of order $n$, which may be called $R^n$, be written parametrically

$$x_i = a_i t^n + b_i t^{n-1} + c_i t^{n-2} + \cdots \qquad (i = 0, 1, 2). \tag{1}$$

If (1) is cut by two lines

$$(\xi x) = \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 = 0, \tag{2}$$

and

$$(\eta x) = \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 = 0, \tag{3}$$

the results of this operation are the two binary $n$-ics

$$(a\xi)t^n + (b\xi)t^{n-1} + (c\xi)t^{n-2} \cdots = 0, \tag{4}$$

* Presented to the Society, September 6, 1910.

and

$$(a\eta)t^n + (b\eta)t^{n-1} + (c\eta)t^{n-2} \cdots = 0, \tag{5}$$

which yield the $n$ parameters of the points in which the lines (2) and (3) cut the curve.

The combinants of (4) and (5) are expressible as rational functions of the two-rowed determinants of the matrix

$$\begin{Vmatrix} (a\xi) & (b\xi) & (c\xi) & (d\xi) \cdots \\ (a\eta) & (b\eta) & (c\eta) & (d\eta) \cdots \end{Vmatrix}. \tag{6}$$

But if in any two-rowed determinant of (6), the quantities $x_0$, $x_1$, $x_2$ are substituted for the coordinates of the point in which (2) and (3) intersect, the result may be expressed as a three-rowed determinant. For instance,

$$\begin{vmatrix} (a\xi) & (b\xi) \\ (a\eta) & (b\eta) \end{vmatrix} = \begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix} = |abx|; \tag{7}$$

and evidently the other determinants of (6) assume similar form. Consequently, by means of the above translation scheme * from the combinants of (4) and (5) certain loci are derived which are related to the $R^n$ in a special manner. Certain sets of points on the $R^n$ defined by some projective relation are connected in this way with a point of the plane, namely, the intersection of (2) and (3). This fact alone warrants the assertion that loci related to the $R^n$ in such a manner are *covariant curves* † of the $R^n$. But it is important to give an actual algebraic proof of this fact.

In order to prove formally that these curves are covariants, it is sufficient to show that they are unaltered by any linear transformation of the $x$'s or $t$'s.

Let the transformation of the $x$'s be

$$x_i' = l_i x_0 + m_i x_1 + n_i x_2 \qquad (i = 0, 1, 2). \tag{8}$$

After this transformation (1) assumes the form

$$x_i' = (l_i a_0 + m_i a_1 + n_i a_2)t^n + (l_i b_0 + m_i b_1 + n_i b_2)t^{n-1} \cdots, \tag{9}$$

or

$$x_i = (l_i a)t^n + (l_i b)t^{n-1} + (l_i c)t^{n-2} \cdots, \tag{10}$$

where $(l_i a) = l_i a_0 + m_i a_1 + n_i a_2$, etc.

* GRACE and YOUNG, *Algebra of Invariants*, pp. 314–318.

† GRACE and YOUNG in the place already referred to have practically proved that such loci are covariant.

From what has been stated the combinant curves of (10) are expressible as sums of terms, each term being of the same degree in determinants of the type

$$(11) \qquad \begin{vmatrix} (l_0 a) & (l_0 b) & (l_0 x) \\ (l_1 a) & (l_1 b) & (l_1 x) \\ (l_2 a) & (l_2 b) & (l_2 x) \end{vmatrix}.$$

But by the theorem for the multiplication of determinants (11) becomes

$$(12) \qquad |lmn|\,|abx|;$$

also each three-rowed determinant of the type (11) formed from (10) is equal to the product of $|lmn|$ by the corresponding determinant formed from (1). Hence, as $|lmn|$ factors out of each term of the combinants of (10) to the same degree, these curves are unaltered by a transformation of the $x$'s. This is a property which is possessed by any homogeneous function of the determinants of the type $|abx|$; however, not every such function is a covariant. But if such a function is also unaltered by a linear fractional transformation of the parameter $t$, then such a function is a covariant. Therefore it remains to show that combinant curves possess also this property.

Let the transformation of the $t$'s ($t = t_1/t_2$) be

$$(12) \qquad t_1 = m_1 t_1 + m_2 t_2, \qquad t_2 = l_1 t_1 + l_2 t_2.$$

Transforming (1) by means of (12) we obtain

$$(13) \qquad x_i = a_i' t^n + b_i' t^{n-1} + c_i' t^{n-2} \cdots \qquad (i = 0, 1, 2).$$

If we cut (13) by the two lines (2) and (3) we shall obtain two binary $n$-ics,

$$(14) \qquad (a'\xi) t^n + (b'\xi) t^{n-1} \cdots = 0,$$

and

$$(15) \qquad (a'\eta) t^n + (b'\eta) t^{n-1} \cdots = 0.$$

But (14) and (15) are exactly what we should have obtained by operating upon (4) and (5) with (12). However, the combinants of (4) and (5) are unaltered by any such transformation as (12); hence *the combinant curves are unaltered by a transformation of the parameter t and are covariant curves of the* $R^n$. All loci obtained in the manner indicated are covariants, but other covariants occur which cannot be derived as combinants.

Salmon* has discussed the combinants of two binary quartics from an algebraic standpoint and we shall apply his result with some extensions to a study of those covariants of the $R^4$ which can be derived in this manner.

* *Higher Algebra*, third edition, pp. 200–206.

### § 1b. *Covariant Curves of $R^4$ as Combinants.*

Let the equation of the $R^4$ be written parametrically

$$x_i = a_i t^4 + 4b_i t^3 + 6c_i t^2 + 4d_i t + e_i \qquad (i=0, 1, 2). \tag{16}$$

If the $R^4$ is cut by the two lines (2) and (3) we obtain the two binary quartics,

$$U = (a\xi)t^4 + 4(b\xi)t^3 + 6(c\xi)t^2 + 4(d\xi)t + (e\xi) = 0, \tag{17}$$

$$V = (a\eta)t^4 + 4(b\eta)t^3 + 6(c\eta)t^2 + 4(d\eta)t + (e\eta) = 0, \tag{18}$$

giving the parameters of the points in which (2) and (3), respectively, cut the curve. Considering the $u$ and $v$ above to be the same as those of Salmon * by reason of (5) and (7) we have

$$\begin{aligned} &\alpha=(ab)=|abx|, \quad \alpha'=(de)=|dex|, \quad \beta=(ad')=|adx|, \quad \beta'=(be')=|bex|, \\ &\gamma=(ae')=|aex|, \quad \delta=(bd')=|bdx|, \quad \lambda=(ac')=|acx|, \quad \lambda'=(ce')=|cex|, \\ &\qquad u=(be')=|bcx|, \quad u'=(cd')=|cdx|. \end{aligned} \tag{19}$$

Consider the pencil $u + Kv$ as a single binary quartic, substitute its coefficients in its two invariants $g_2$ and $g_3$, and let the resulting quadratic and cubic in $K$ be written

$$S_0' K^2 + S_1' K + S_2' = 0 \qquad \text{and} \qquad T_0 K^3 + T_1 K^2 + T_2 K + T_3 = 0; \tag{20}$$

the quadratic gives those two values of $K$ for which $u + Kv$ becomes a self-apolar quartic; and the cubic those values of $K$ for which $u + Kv$ becomes a catalectic quartic.

Equating to zero the discriminant of the quadratic of (20) and making the substitutions (7), we have

$$\begin{aligned} A \equiv |aex|^2 + 16|bdx|^2 + 12|acx||cex| - 48|bex||cdx| - 8|abx||dex| \\ -8|adx||bex| = 0, \end{aligned} \tag{21}$$

which is the locus whose tangents cut $R^4$ in self-apolar sets of points. If the $g_2$ of (4) were formed it would be the envelope of lines which cut $R^4$ in self-apolar quartics. Hence $A$ is the point equation of the conic $g_2$. A line which cuts $R^4$ in three consecutive points is a line of $A$, for the $g_2$ of a quartic having a triple root vanishes. Consequently the six inflexional tangents of $R^4$ touch $A$, and indeed $A$ is often spoken of as the conic on the six inflexional tangents of $R^4$.

The condition for $u$ and $v$ to have the same apolar cubic, or that each may be expressed linearly in the same three fourth powers, leads to a second combinant which with the substitutions (7) becomes

* *Higher Algebra*, third edition, pp. 200–206.

$$B \equiv |acx||cex| - |bcx||cdx| - |adx||cdx| - |bex||bcx| + |bdx|^2 - |abx||dex| = 0. \tag{22}$$

This conic may be identified with the conic which Stahl* gives parametrically; it is the locus of the vertices of flex-triangles of all first osculants.

The eliminant of $u$ and $v$ expresses the condition for a common root or the condition that they intersect on the curve. Hence by reason of (7) $R$ becomes the point equation of the $R^4$.

The Jacobian of $u$ and $v$ is

$$\alpha t^6 + 3\lambda t^5 + (3\beta + 6\mu)t^4 + (\gamma + 8\delta)t^3 + (3\beta' + 6\mu')t^2 + 3\lambda' t + \alpha' = 0. \tag{23}$$

By a well known property of the Jacobian (23) gives the six tangents from a point to $R^4$. The condition that these form a self-apolar set is

$$-40I_2\dagger = A + 48B. \tag{24}$$

Consequently the locus of points such that tangents from them to $R^4$ form self-apolar sets is the conic

$$A + 48B = 0. \tag{25}$$

Further we find that the following relation holds

$$320(I_2^2 - 100I_4) = (A - 16B)^2 - R, \tag{26}$$

where $I_2$ and $I_4$ are the well known invariants of the sextic (23). But $I_2^2 - 100I_4 = 0$ for a sextic which has either a triple root, or a double root and the other four forming a self-apolar set. Consequently such points lie on the quartic curve

$$(A - 16B)^2 - R = 0. \tag{27}$$

The points on $R$ which behave in this manner are on the conic $A - 16B = 0$. Hence the six flexes and the two points $q_i$ (which have the property that tangents drawn from them to $R^4$ form self-apolar sets) lie on the conic

$$A - 16B = 0. \tag{28}$$

It is worthy of notice that the above not only gives an easy way to write down the equation of the conic on the flexes but is an independent analytical proof that they do lie on a conic with the two points $q_i$.

Salmon's combinant equation $C = 0$, expressing the condition that a member of the pencil $u + Kv$ can have two squared factors, and from our standpoint the locus of points such that lines drawn from them cut the $R^4$ in quartics which

* W. STAHL, Journal für die reine und angewandte Mathematik, vol. 101 (1886).

† $I_2 = 0$ is the self-apolarity condition of (23).

have two squared factors, can be nothing else than the point equation of the four double tangents of the $R^4$. Since it may be written

$$128\,C = (A - 48B)^2 - R, \tag{29}$$

it is evident that the conic on the eight points of contact of the four double tangents is

$$A - 48B = 0. \tag{30}$$

Forming the Hessian of the cubic of (20) and requiring it to be apolar to the quadratic of (29) yields a combinant which may be expressed thus,

$$128I = R - (A - 16B)(A + 48B). \tag{31}$$

When $I \equiv 0$, the $R^4$ breaks up into two conics. If the two lines from a point which cut the $R^4$ self-apolarly are apolar to the Hessian pair of the three lines which cut $R^4$ in catalectic quartics, then $I$ vanishes; therefore together with (7) it gives the locus of such points.

Forming the third transvectant of $u$ and $v$ we get a quadratic in $t$ whose discriminant vanishes on the locus

$$A - 12B = 0. \tag{32}$$

This may be identified with Stahl's conic $N$. The first osculant of $R^4$ at a point $t'$ is an $R^3$ and therefore has three flexes on a line. The equation of this flex-line is a quadratic in $t'$ and the discriminant of this quadratic may be identified with (32). Or writing the $R^4$ symbolically

$$(a\xi)(\alpha t)^4 = 0 \qquad \text{and} \qquad (b\xi)(\beta t)^4 = 0, \tag{33}$$

then taking the third transvectant we have

$$|abx||\alpha\beta|^3(\alpha t)(\beta t) = 0, \tag{34}$$

which is the conic $N$ given parametrically. Of course the letters in (33), (34) are only symbols and must not be confused with the same letters already used with different meanings.

The eliminant of the cubic and quadratic of (20) is the condition for such a $K_1$ to exist that $u + K_1 v = 0$ has roots that are both self-apolar and harmonic pairs. But a quadratic for which $g_2 = g_3 = 0$ has a triple root, and therefore with (7) Salmon's $D = 0$ becomes the equation of the six flex tangents of the $R^4$.

The combinant $E$, the discriminant of the cubic of (20), evidently yields the locus of point whose lines cut $R^4$ in catalectic quartics. It is therefore the point equation of the $g_3$ of equation (4).

Suppose that the polar of the quadratic of (20) as to the cubic of (20) is $\phi_1$, and the polar of $\phi_1$ as to the quadratic is $\phi_2$, by using the cubicovariant of the

cubic instead of the cubic itself we obtain $\phi_1'$ and $\phi_2'$ by the same operations. The condition for $\phi_1 = 0$ and $\phi_2'$ (or $\phi_1'$ and $\phi_2$) to be the same is the combinant $M$, and by using (7) we have the locus of such points.

Before leaving covariant curves we observe that from Salmon's identities

$$D - E = -2IB - B^2(A - 16B), \tag{35}$$

and

$$128I = R - (A + 48B)(A - 16B), \tag{36}$$

it follows that

$$64(D - E) = B[(A - 16B)^2 - R]. \tag{37}$$

Consequently,* *twelve of the intersections of the flex tangents of $R^4$ and $E$ lie on the conic $B$.*

Also from (26) and (37) we have

$$320(D - E) = B[(A + 48B)^2 - 100I_4]. \tag{38}$$

If $I_4 \equiv 0$ (i. e. if tangents drawn from any point to $R^4$ form a catalectic set), the intersections of $D$ and $E$ lie on two conics; the flex tangents touch $E$ along $A + 48B = 0$ and intersect $E$ along $B$. This particular $R^4$ is of the lemniscate type.

§ 2. *Invariants of $R^4$ as Combinants of the Fundamental Involution.*

If the $R^4$ be referred to a special triangle of reference its equations are

$$\begin{aligned} x_0 &= at^4 + 4bt^3, \\ x_1 &= 4dt + e, \\ x_2 &= 4bt^3 + 6ct^2 + 4dt. \end{aligned} \tag{1}$$

The condition for $R^4$ to have a triple point,† or the condition for $R^4$ to have a perspective point for (1) becomes

$$B_1' \equiv 36a^2c^2e^2 + 256ab^2d^2e - 96a^2cd^2e - 96ab^2ce^2 - 16a^2bde^2 = 0. \tag{2}$$

If the $g_2$ of $u$ of Section 1 is found for (1), its discriminant multiplied by eight may be written

$$A_1' = -6a^2c^2e^2 - 96b^2c^2d^2 + 8a^2bde^2 + 48abc^2de - 32ab^2d^2e = 0. \tag{3}$$

Since the six inflexional tangents are lines of the conic $g_2$, $A'$ vanishes when the $R^4$ has three concurrent flex tangents.

*See THOMSEN, American Journal of Mathematics, vol. 32 (1910), p. 222.

†W. STAHL, Mathematische Annalen, vol. 38 (1891); GROSS, Mathematische Annalen, vol. 32 (1888).

Calculating the conic on the flexes from the preceding section for (1) and taking its discriminant, we have

$$C'_1 \equiv 256(a^3b^2cd^4e^2)^* - 64(a^4c^2d^4e^2) + 32(a^4c^3d^2e^3)$$
$$(4) \qquad - 192a^3b^2c^2d^2e^3 - 512a^2b^3c^2d^3e^2 + 128(a^3bc^3d^3e^2)$$
$$- 32a^3bc^4de^3 - 4a^4c^4e^4 = 0.$$

Since it is known that if an $R^4$ has three flexes on a line it has a fourth, $C'=0$ is the condition for four collinear flexes of the $R^4$.

These three invariants are independent, as may be shown in the following manner: $A'$ and $B'$ are evidently independent. Further if $C'$ were expressible in terms of $A'$ and $B'$, such a relation as $C' = \lambda B'^2 + \mu A'B'$ would have to hold because $C'$ contains no term in $b^4c^4d^4$, i. e., $B'$ would have to be a factor of $C'$, but this is disproved by making $b=0$ in (2), (3) and (4). Similarly $D'$ is shown to be independent of $A'$, $B'$, and $C'$.

The condition for $A - 16B$ in lines to be apolar to $A$ in points we shall call $R'_2 = 0$, and we find that

$$(5) \qquad 36R'_2 + 108C'_1 + B'^2_1 + 10A'_1B'_1 + 16A'^2_1 = 0.$$

If $H=0$ is the apolarity condition of $A$ in lines to $A - 16B$ in points, then

$$(6) \qquad 6H + A'_1B'_1 + 8A'^2_1 = 0.$$

The discriminants of $A$ and $B$ are known, and therefore, the discriminant of the whole pencil

$$(7) \qquad \lambda(A - 16B) + \mu A = 0$$

is

$$(8) \qquad C'_1\lambda^3 + R'_2\lambda^2\mu + H\lambda\mu^2 - A'^2_1\mu^3 = 0.$$

But a more convenient formula than (8) is the discriminant of

$$(9) \qquad \lambda'(cA + dB) + \mu'(aA + 16bB) = 0,$$

which is

$$\begin{aligned}
&\big[-(c+d)^3A'^2_1 - d(c+d)^2H + d^2(c+d)R'_2 - d^3C'_1\big]\lambda^3 \\
&+\big[-3(a+b)(c+d)^2A'^2_1 - \big(b(c+d)^2 + 2d(a+b)(c+d)\big)H \\
&\qquad + \big(d^2(a+b) + 2bd(c+d)\big)R'_2 - 3bd^2C'_1\big]\lambda^2\mu' \\
&+\big[-3(a+b)^2(c+d)A'^2_1 - \big(2(a+b)(c+d)b + (a+b)^2d\big]H \\
&\qquad + \big(2bd(a+b) + (c+d)b^2\big)R'_2 - 3b^2dC'_1\big]\lambda'\mu'^2 \\
&+\big[-(a+b)^3A'^2_1 - (a+b)^2bH + (a+b)b^2R'_2 - b^3C'_1\big]\mu'^3 = 0.
\end{aligned} \qquad (10)$$

* The expressions in parentheses carry with them their conjugate expressions.

By substituting proper values in (9) and (10) any invariant of any member of the pencil of conics may be easily found. For instance, by a method to be given presently the discriminant of $B$* is found to differ only by a numerical factor from the undulation condition. Hence by substitution in (9) and (10) the undulation condition is

$$R_2' - H - A_1'^2 - C_1' = 0, \tag{11}$$

or by reason of (5) and (6)

$$144\,C_1' + (B_1' + 2A_1')^2 = 0. \tag{12}$$

Also the invariant whose vanishing is the condition for three concurrent double tangents is the square root of the discriminant of the conic $A - 12B$, or at least it differs only by a factor. Hence the condition for $A - 12B$ to degenerate is

$$-(B_1' + 6A_1')^2 = 0, \tag{13}$$

and for three concurrent double tangents,

$$B_1' + 6A_1' = 0. \tag{14}$$

The conic $A - 16B$ meets $R^4$ in two points $q_i$ besides the six flexes. By substituting (1) in $A - 16B$ and comparing with the sextic giving the flexes we find a quadratic giving the $q$'s. Its discriminant is

$$B_1' + 8A_1' = 0; \tag{15}$$

hence (15) is the condition for the $q$'s to unite.

Having taken the $R^4$ in a workable form to find relations among its invariants, we shall now give the most important invariants in their most general forms. It is well known that all line sections of the $R^4$ are apolar to a pencil of quartics which constitute the *Fundamental Involution*. From the theorem of Grassmann† on the proportionality of determinants it follows that the combinants of the Fundamental Involution are invariants of the $R^4$. An easy method to pass from the combinants of line sections of an $R^4$ to combinants of the Fundamental Involution is to replace each determinant by its complementary, i. e., $|acx|$ and $|bkx|$ must be replaced by $|bde|$ and $|ace|$, respectively, etc. Of course, allowance must be made for binomial coefficients. For instance, if these changes are made in the combinant $B$ of Section 1*b* we have

$$\begin{aligned} B' \equiv\ & |bde||abd| - |ade||abe| - |bce||abe| \\ & - |acd||ade| + |ace|^2 - |cde||abc| = 0, \end{aligned} \tag{16}$$

* This was proved in a different way by Dr. THOMSEN in his dissertation, Johns Hopkins University, 1909.

† W. F. MEYER, *Apolarität und Rationale Curven*, § 11.

which is the triple-point condition for $R^4$ written without binomial coefficients. Further, if these changes are made in $A$ of Section 1*b* the result is $-768$ times the discriminant of $A$ for $R^4$ written without binomial coefficients.

We shall summarize the results of these two sections in a table. Let $(R^4)$ mean the $R^4$ written parametrically with binomial coefficients and $\check{(}R^4\check{)}$ mean $R^4$ written in the same way without them. Also let $A'$ stand for the combinant $A$ of the Fundamental Involution which may be derived from $A$ as already explained. In the first column is given the name of the combinant of two line sections of $(R^4)$; in the second we have the locus resulting from the process explained in Section 1*a*; in the third is the same combinant of the Fundamental Involution giving an invariant of $\check{(}R^4\check{)}$ whose meaning appears in the fourth column. Also if $A'$ means an invariant of $\check{(}R^4\check{)}$ we shall understand that $A_1'$ stands for the same invariant of $(R^4)$, possibly differing by a factor. Evidently any such invariant as $A_1'$ can be obtained from $A'$ by substitution of binomial coefficients, i. e., $|cde|$ and $|bcd|$ of $A'$ would be replaced by $24|cde|$ and $96|bcd|$, respectively, to obtain a multiple of $A_1$.

*Table of Related Invariants and Covariants.*

| Combinant of two line sections of $(R^4)$. | Covariant Curve of $(R^4)$. | Combinant of F. I. giving an invariant of $\check{(}R^4\check{)}$. | Condition for $\check{(}R^4\check{)}$ to have |
|---|---|---|---|
| $A$ | Locus whose lines cut $(R^4)$ in self-apolar sets. | $A'$ | 3 concurrent flex-tangents. |
| $B$ | Locus of vertices of flex $\Delta$'s of 1st osculants. | $B'$ | Triple point. |
| $C$ | Product of 4 double tangents. | $C'$ | 4 collinear flexes. |
| $D$ | Product of 6 flex tangents. | $D'$ | Cusp. |
| $R$ | Point equation of $(R^4)$. | $R'$ | Undulation. |
| $E$ | Locus whose lines cut $(R^4)$ in catalectic sets. | $E'$ | Tac-node. |
| $A-12B$ | Locus of flex lines of 1st osculants. | $A'-12B'$ | Its 2 $q$'s unite. |
| $M$ | Remote meaning. | $M'$ | Skew invariant. |

For the purpose of deriving these invariants a special form of the $R^4$ has been used; and after they have been derived, to identify them with the proper combinant of the Fundamental Involution is a comparatively easy matter. This was the case with $A'-12B'$, which as a combinant of the Fundamental Involution one would hardly have expected to give the condition for three concurrent

double tangents. On the other hand, forming the cubic of (8), Section 1*a*, of the Fundamental Involution, we have three catalectic sets associated with the double points; if two of these are the same we have a tac-node, the condition for which is $E' = 0$, and this would probably be a difficult result to obtain directly from the equations of the curve. $D' = 0$ is the condition for $(R^4)$ to have a doubly perspective quintic, and therefore the condition for a cusp.

Let the invariants $A'$, $B'$, $C'$, and $D'$ be called $I_2$, $I'_2$, $I_4$, and $I_6$, respectively. *These four invariants constitute a complete system in the sense that any other invariant relation on the $R^4$ can be expressed in terms of them*, i. e. any other invariant of the $R^4$ is connected with these four by an algebraic relation. The proof of this fact depends upon a theorem due to Stroh,* which applied here says that every combinant of two binary quartics multiplied by the proper power of an invariant (which in this case is the undulation condition) is an invariant of a binary sextic.† In this case $I_2(R^2)^2$, $I'_2(R'^2)^2$, $I_4(R'^2)^4$, $I_6(R'^2)^6$, and $I_k(R'^2)^k$ (where $k$ is the degree of an invariant) become $J_6$, $J'_6$, $J_{12}$, $J_{18}$, and $J_{3k}$ of a binary sextic. But $I_2$, $I'_2$, $I_4$, and $I_6$ are independent, hence $J_0$, $J_6$, $J_{12}$, and $J_{18}$ are independent. Suppose then that there is an invariant $I_k$ of the $R^4$ which gives rise to an $I_{3k}$ of the binary sextic. Then we should have five invariants of a binary sextic, four of which are independent, and these are always connected by an algebraic relation. Consequently any $I_k$ of the $R^4$ is algebraically expressible in terms of the four invariants $I_2$, $I'_2$, $I_4$, and $I_6$.

## § 3. *The $R^4$ as Plane Sections of the Steiner Quartic Surface.*

The invariants of the $R^4$ may be treated very neatly by considering the $R^4$ as plane sections of the Steiner Quartic Surface‡ which we shall call $S^4$.

The point equation of the $S^4$ referred to its trope planes is

$$\sqrt{x_4} = \sqrt{x_0} + \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} = 0; \tag{1}$$

we propose to cut this by the plane

$$(\alpha x) = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0; \tag{2}$$

in this manner any relation among the $\alpha$'s is of (2) yields a special $R^4$, or the invariants of the $R^4$ may be expressed in terms of the symmetric functions of the $\alpha$'s of equation (2).

For instance, any plane through the point (1111) cuts out an $R^4$ with a triple point and therefore the condition for an $R^4$ with a triple point is that (2) be on

* E. Stroh, Mathematische Annalen, vol. 34 (1889), pp 321-323.

† W. Stahl, Crelle's Journal, vol 104 (1889), p. 302.

‡ The Steiner Quartic Surface may be obtained as the polar of a plane as to a tetrahedron. (1) is the polar of the plane (1111) as to $\xi_0\xi_1\xi_2\xi_3 = 0$. It is of the third class, fourth order, with an enveloping cone of order six. Hence a plane section is a curve of order four and class six—an $R^4$.

the point (1111), which is $\alpha_0+\alpha_1+\alpha_2+\alpha_3=0$, and this as a symmetric function of the $\alpha$'s is

$$S_1=0. \tag{3}$$

Any plane on a pinch point cuts out an $R^4$ with a cusp; the six pinch points have the coordinates (1100), $\cdots$, (0011), hence we have an $R^4$ with a cusp when any one of the six factors of

$$(\alpha_0+\alpha_1)(\alpha_0+\alpha_2)(\alpha_0+\alpha_3)(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)=0 \tag{4}$$

vanishes. Expressed in terms of symmetric functions, (4) becomes

$$S_1S_2S_3-S_3^2-S_1^2S_4=0, \tag{5}$$

which is the general condition for a cusp. Similarly when any two factors of (4) vanish we have two cusps, and by taking the product of the factors of (4) five at a time we obtain the condition for a second cusp after (5) has vanished. I find this to be

$$2S_1S_2^2+S_2S_3+S_1^2S_3-S_1S_4=0. \tag{6}$$

By the same argument, after (5) and (6) have vanished the condition for a third cusp is

$$S_1S_3+32S_4+S_2^2+2S_1^2S_2=0. \tag{7}$$

If (2) is a tangent plane of (1), the point of tangency is a double point in the curve of the section. Nodes also occur where (2) cuts the double lines of the $S^4$. The condition for an extra node is that (2) be a plane of (1). The plane equation of (1) is

$$(1/\xi_i)=1/\xi_0+1/\xi_1+1/\xi_2+1/\xi_3=0. \tag{8}$$

and hence the condition for an extra node, or for $R^4$ to degenerate, is

$$S_3=0. \tag{9}$$

The points where (2) cuts the trope conics are points of contact of the double tangents in the curve of section or the double tangent are the intersections of (2) and trope planes. Hence if (2) is on a vertex of the reference tetrahedron the corresponding $R^4$ has three concurrent double tangents. As this can occur in four ways, the condition for three concurrent double tangents is

$$S_4=0. \tag{10}$$

Evidently this could not occur twice unless (9) holds, in which case $R_4$ degenerates.

The line joining the two points where (2) cuts a trope conic is a double tan-

gent; if these two points come together the intersection of (2) and a trope plane becomes a line of the corresponding trope conic and in the section this point is an undulation. Hence the condition for an undulation is that the intersection of (2) and a trope plane be a line of the corresponding trope conic. This may occur in four ways and the general undulation condition is the product

$$\prod^{4}(\alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2) = 0,$$

or

$$S_2 S_3^2 - S_1 S_3 S_4 + S_4^2 = 0. \tag{11}$$

By the same argument as in case of cusps, the condition for a second undulation after (11) has vanished is found to be

$$S_3^2 + S_1 S_2 S_3 - S_1^2 S_4 + 7 S_2 S_4 = 0. \tag{12}$$

Also for a third, after (11) and (12) have vanished we have

$$S_2^2 + S_1 S_3 + 2 S_4 = 0. \tag{13}$$

By the use of a result of Richmond and Stuart * the quadric associated with the $S^4$ containing all those points which in plane sections are flexes is found in our notation,

$$8(\alpha_0^3 x_0^2 + \alpha_1^3 x_1^2 + \alpha_2^3 x_2^2 + \alpha_3^3 x_3^2) - S_3(\Sigma x_0^2 - 2\Sigma x_0 x_1) = 0. \tag{14}$$

By its form the quadric in the second parenthesis touches the edges of the tetrahedron at their mid-points, and in sections gives the points of contact of the double tangents. Knowing the meaning of (14) and of the quadric just mentioned and recalling that they (or the corresponding conics in Section 1*b*) are proportional to $A - 16B$ and $48\ B - A$, respectively, we see that the quadric $\Sigma\alpha_0^3 x_0^2 = 0$ is the locus of points which in plane sections yields the conic which is the envelope of flex-lines of all first osculants, as it is proportional to $A - 12B$.

It is desirable to form the discriminant of the pencil of conics as has been done in Section 2. In order to do this in a way that involves the $\alpha$'s symmetrically, we find the plane equation of (14) and substitute the $\alpha$'s for $\xi$'s. Thus by using a result of Salmon † after dividing out by $S_3$, the discriminant of

$$\mu\Sigma\alpha_0^3 x_0^2 - \lambda S_3(\Sigma x_0^2 - 2\Sigma x_0 x_1) = 0 \tag{15}$$

is exhibited as follows,

$$S_4^2\mu^3 - (S_2 S_3^2 - 5 S_1 S_3 S_4 + 8 S_4^2)\mu^2\lambda + 4(S_1^2 S_3^2 - 2 S_2 S_3^2 - S_1 S_3 S_4)\mu\lambda^2 + 4 S_3^2(S_1^2 - 4 S_4)\lambda^3 = 0; \tag{16}$$

* Proceedings of the London Mathematical Society, series 2, vol. 1 (1904), pp. 129–132.

† *Geometry of Three Dimensions;* p. 50, § 67, p. 58, § 79.

and (16) in the notation of Section 1*b* is the discriminant of

$$\mu(A - 16B) + \lambda(A - 48B) = 0. \tag{17}$$

Equating (17) to $A$ and substituting in (16) we find the discriminant of $A$ to be

$$A_2 \equiv [\tfrac{2}{3}(S_1 S_3 - 4S_4)]^2 = 0, \tag{18}$$

therefore three flex tangents are concurrent if

$$S_1 S_3 - 4S_4 = 0. \tag{19}$$

Or by equating (17) to $B$ we find the discriminant of $B$

$$\frac{1}{4.36^2}(S_4^2 + S_2 S_3^2 - S_1 S_3 S_4) = 0, \tag{20}$$

but (20) is only a multiple of (11), showing that $B$ degenerates when $R^4$ has an undulation.

If we identify (17) with

$$\nu A + K(A - 16B) = 0 \tag{21}$$

we find that the discriminant of (21) is

$$\begin{aligned}[\tfrac{2}{3}(S_1 S_3 - 4S_4)]^2\nu^3 + \tfrac{4}{27}(S_1^2 S_3^2 - 20S_1 S_3 S_4 + 64S_4^2)\nu^2 K \\ + \tfrac{4}{81}(64S_4^2 - 5S_1^2 S_3^2 + 12S_2 S_3^2 + 4S_1 S_2 S_4)\nu K^2 \\ + \tfrac{4}{81}(8S_1 S_3 S_4 + S_1^2 S_3^2 - 4S_2 S_3^2)K^3 = 0.\end{aligned} \tag{22}$$

Let us write equation (22) briefly as

$$A_2^2\nu^3 + H_2\nu^2 K + R_3\nu K^2 + C_2 K^3 = 0. \tag{23}$$

The importance of (23) will appear later.

If a point where a double line of $S^4$ cuts (2) is moved up to be a point of (14) we have brought a node and a flex together and have a flecnode; except that at the two pinch points which are on each double line we get cusps.

The points where the double lines of $S^4$ cut (2) are

$$\begin{aligned} x_0 = x_1 = \alpha_2 + \alpha_3, \quad & x_2 = x_3 = -(\alpha_0 + \alpha_1); \\ x_0 = x_2 = \alpha_1 + \alpha_3, \quad & x_1 = x_3 = -(\alpha_0 + \alpha_2); \\ x_0 = x_3 = \alpha_1 + \alpha_2, \quad & x_1 = x_2 = -(\alpha_0 + \alpha_3). \end{aligned} \tag{24}$$

Substituting these in equation (14) and factoring the cusp condition twice out each time, I find that the conditions under which a flecnode can occur are three expressions of the type

$$(\alpha_0 - \alpha_1)^2(\alpha_2 + \alpha_3) + (\alpha_2 - \alpha_3)^2(\alpha_0 + \alpha_1) = 0; \tag{25}$$

i. e., the vanishing of any one of the three expressions of the type of (25) insures a flecnode.

By arguments similar to those already used the general condition for a flecnode is shown to be

$$32S_1S_2S_3^2 - 4S_1^2S_2^2S_3 - 64S_3^3 + S_1^4S_2S_3 - 5S_1^3S_2^2 + 16S_1^2S_3S_4 - S_1^5S_4 = 0. \tag{26}$$

The condition for a second flecnode after (26) is satisfied is

$$S_1^2S_2^2 - 16S_1S_2S_3 + 48S_3^2 + S_1^3S_3 - 48S_1^2S_4 = 0. \tag{27}$$

Also, for a third, in addition to (26) and (27) we find

$$S_1S_2 - 6S_3 = 0. \tag{28}$$

Obviously an $R^4$ cannot have three flecnodes without one of them being a biflecnode. The $R^4$ is cut in eight points by $A - 16B = 0$; six of these points are flexes and the other two are the points $q$. For a flecnode, a $q$ and a flex must come together; this could occur only twice and a third flecnode would require two flexes of the $R^4$ to cross, thus forming a biflecnode.

The matter of biflecnode conditions is somewhat unsatisfactory. To find them we introduce a conic of fundamental importance. The conic

$$\lambda(x_0^2 + x_1^2 + x_2^2) + 2fx_1x_2 + 2gx_0x_2 + 2hx_0x_1 = 0 \tag{29}$$

may be transformed by $x_i = 1/x_i$ into the $R^4$ referred to its nodal triangle, and its equation is

$$\lambda(x_1^2x_2^2 + x_0^2x_2^2 + x_0^2x_1^2) + 2fx_0^2x_1x_2 + 2gx_1^2x_0x_2 + 2hx_2^2x_0x_1 = 0. \tag{30}$$

It may be shown that the invariants of the $R^4$ calculated directly from equation (30) are connected with the invariants we have found from the $S^4$ by the following relations

$$\begin{aligned} 2\lambda &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = S_1, \\ 2f &= \alpha_0 + \alpha_1 - \alpha_2 - \alpha_3, \\ 2g &= \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3, \\ 2h &= \alpha_0 - \alpha_1 - \alpha_2 + \alpha_3. \end{aligned} \tag{31}$$

The equation of a pair of nodal tangents may be written down from (30). If we impose upon each of these tangents the condition that it cut the $R^4$ in three consecutive points the results are the conditions for a biflecnode. These operations yield three pairs of equations of the type

$$h\lambda - gf + g\gamma = 0, \qquad gf - h\lambda + g\gamma = 0, \tag{32}$$

where $\lambda = 0$ is the triple-point condition and $\gamma = 0$ is the cusp condition. Ruling out the vanishing of $\lambda$ or $\gamma$, one such pair as (32) will vanish if only any two of the quantities $(f, g, h)$ vanish. A pair of conditions expressible in symmetric functions of the $\alpha$'s which are equivalent to these two conditions are

$$\sum^{3} f^2 g^2 = 0 \qquad \text{and} \qquad f^2 g^2 h^2 = 0, \tag{33}$$

and these are expressible as

$$3S_1^4 - 16S_1^2 S_2 + 16S_2^2 + 16S_1 S_3 - 64S_4 = 0, \tag{34}$$

and

$$S_1^6 - 8S_1^4 S_2 + 16S_1^2 S_2^2 + 48S_2^3 + 16S_1^3 S_3 - 64S_1 S_2 S_3 + 64S_3^2 = 0. \tag{35}$$

It may be easily verified that the conic on the flexes degenerates when (34) and (35) vanish simultaneously.

# AN APPLICATION OF MOORE'S CROSS-RATIO GROUP TO THE SOLUTION OF THE SEXTIC EQUATION*

BY

ARTHUR B. COBLE†

## *Introduction.*

By making use of the fact that all the double ratios of $n$ things can be expressed rationally in terms of a properly chosen set of $n-3$ double ratios, E. H. MOORE‡ has developed the "cross-ratio group," $C_{n!}$, a Cremona group in $S_{n-3}$ of order $n!$ which is isomorphic with the permutation group of $n$ things. H. E. SLAUGHT§ has discussed the $C_{5!}$ in considerable detail. Both $C_{5!}$ and $C_{6!}$ have been listed by S. KANTOR.‖

I have already pointed out¶ that $C_{n!}$ defines a form-problem which I shall denote by $PC_{n!}$. The solution of $PC_{n!}$ carries with it the solution of the $n$-ic equation, and I have worked out in detail the application to the quintic equation. After the adjunction of the square root of the discriminant of the $n$-ic, a new form-problem, $PC_{n!/2}$, can be enunciated whose underlying group is the even subgroup $C_{n!/2}$ of $C_{n!}$. It appears in the particular case, $n=5$, that the solution of $PC_{5!/2}$ can be expressed rationally in terms of the solution of KLEIN'S $A$-problem.** The transition from the one problem to the other is accomplished by a very direct process which is suggested by simple geometric considerations. Explicit formulæ are always desirable, and these have been lacking hitherto in the solutions of the quintic equation in terms of the $A$-problem. By introducing $PC_{5!/2}$ as an intermediate stage I have obtained such formulæ from the long known invariant theory of the binary quintic.

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† This and allied investigations are being carried on under the auspices of the Carnegie Institution of Washington.

‡ American Journal of Mathematics, vol. 22 (1900), p. 279.

§ American Journal of Mathematics, vol. 22 (1900), p. 343; and vol. 23 (1901), p. 99.

‖ See his *Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene*, Berlin (1895), p. 105, Type XIV; also *Theorie der Transformationen im $R_3$*, etc., Acta Mathematica, vol. 21 (1897), p. 1, in particular, p. 77, Type LI.

¶ These Transactions, vol. 9 (1908), p. 396, cited hereafter as c1.

** An explicit solution of the $A$-problem in terms of the icosahedral irrationality is found in c1, § 5.

It is my object in this paper to determine whether any material advantage is gained in the solution of the sextic equation by the introduction of the form-problem of the cross-ratio group of six things, $PC_{6!}$ or $PC_{6!/2}$. The main result in this direction is found in § 2, namely, that after the adjunction of the square root of the discriminant $\Delta$ of the given sextic $S$, the $PC_{6!/2}$ and therefore $S$ itself has a rational resolvent sextic $\Sigma$. This resolvent $\Sigma$ is of the general diagonal type, i. e., the fifth and third powers of the unknown are missing. This resolvent is not unknown, * and the main point of novelty here is the very natural way in which it is suggested by the $C_{6!}$.

In the first paragraph, the equations of generating substitutions of $C_{6!}$ are derived, particular sets of conjugate points are considered, the invariant spreads of the group are obtained, and the form-problems of $C_{6!}$ and $C_{6!/2}$ are set forth. The solution of the sextic $S$ in terms of the solution of either of these form-problems is exhibited. In the second paragraph the solution of $PC_{6!/2}$ is expressed rationally in terms of the solution of the resolvent $\Sigma$. Some data concerning the history of $\Sigma$ and its connection with a particular sextic $\Phi$ of MASCHKE are reviewed. These data are correlated to some extent by a geometric interpretation. A solution of $PC_{6!}$ by means of hyperelliptic modular functions also appears. In the third paragraph the reduction of the solution of $\Sigma$ to the solution of the form-problem $PG_{360}$ of the VALENTINER collineation group $G_{360}$ in $S_2$ is outlined. The method there used applies equally well to the given sextic $S$,† so that the advantage which arises from the introduction of $\Sigma$ is due only to the absence of two of its coefficients. But the rational functions of the coefficients required for the reduction to $PG_{360}$ are of necessity so complicated ‡ that this advantage is important.

## § 1. *The Invariants and Form Problems of* $C_{6!}$.

A sextic $S$ with given roots $z_1, z_2, \cdots, z_6$ can be transformed linearly in such a way that $z_6$ becomes $\infty$ and the other roots become $y_1, y_2, \cdots, y_5$ respectively, where $\sum_1^5 y_i = 0$. The transformed roots $y$ are determined to within a factor of proportionality. Their values in terms of the differences of the roots $z$ are

$$
\begin{aligned}
\rho y_1 &= 21\ 36\ 46\ 56 + 31\ 26\ 46\ 56 + 41\ 26\ 36\ 56 + 51\ 26\ 36\ 46, \quad ik = z_i - z_k,\\
\rho y_2 &= 12\ 36\ 46\ 56 + 32\ 16\ 46\ 56 + 42\ 16\ 36\ 56 + 52\ 16\ 36\ 46,\\
\rho y_3 &= 13\ 26\ 46\ 56 + 23\ 16\ 46\ 56 + 43\ 16\ 26\ 56 + 53\ 16\ 26\ 46,\\
\rho y_4 &= 14\ 26\ 36\ 56 + 24\ 16\ 36\ 56 + 34\ 16\ 26\ 56 + 54\ 16\ 26\ 36,\\
\rho y_5 &= 15\ 26\ 36\ 46 + 25\ 16\ 36\ 46 + 35\ 16\ 26\ 46 + 45\ 16\ 26\ 36.
\end{aligned} \tag{1}
$$

* See the references at the end of § 2.

† COBLE, Mathematische Annalen, vol. 70 (1911), p. 337; cited hereafter as c2.

‡ An estimate of the number and order of these functions is found in c2.

The differences of the transformed roots $y$ are

$$\rho(y_i - y_k) = -5 \cdot ik\,\overline{l6}\;\overline{m6}\;\overline{n6} \quad (i, k, l, m, n = 1, 2, \cdots, 5). \tag{2}$$

The system of values $y$ subject to the relation, $\sum y_i = 0$, can be interpreted as the homogeneous coördinates of a point $y$ in $S_3$. In this system of coördinates there is associated with the five reference planes, $y_i = 0$, a set of five points, each the polar point of one plane as to the tetrahedron of the other four. Denote these five points by $p_1, p_2, \cdots, p_5$; their coördinates are $p_1(-4, 1, 1, 1, 1), \cdots, p_5(1, 1, 1, 1, -4)$.

In the formulæ (1) let $z_6$ be a variable parameter, $t$. Then the point $y$ describes a rational twisted cubic. As $t$ takes the values $z_1, z_2, \cdots, z_6$, $y$ takes the positions, $p_1, p_2, \cdots, p_5, y$. Thus to construct the point $y$ which corresponds to a given sextic $S$, we associate five of the roots of $S$ to the points $p$ and pass through these points a rational cubic such that their parameters $p$ on the cubic are projective to the five roots of $S$. In this projectivity, the sixth root of $S$ determines a sixth point $y$ on the cubic curve. Since five roots of $S$ can be associated with the five points in 720 ways, a sextic $S$ determines a set of 720 points $y$. The change in $y$ due to a permutation of the roots of $S$ from some given order furnishes a transformation of Moore's group, $C_{6!}$. The 720 points $y$ determined by $S$ are a set of conjugate points under $C_{6!}$.

Let $T_{(\ldots)\ldots(\ldots)}$ denote a permutation written in cycle form from the order $z_1, z_2, \cdots, z_6$ to the order $z_1', z_2', \cdots, z_6'$. The same symbol will be used for the transformation which carries $y$ into $y'$. Clearly the permutations of $z_1, \cdots, z_5$ alone give rise to the permutation group of the five coördinates $y$, i. e., to the collineation group, $C_{120}$, which transforms the system of five planes $y_i$ or the system of five points $p_i$ into itself. The $C_{6!}$ can be generated by this group and an additional transformation, say $T_{(16)}$. To obtain this transformation we make use of formulæ (2).

$$\begin{aligned}
\rho'(y_1' - y_2') &= -5 \cdot \overline{1'2'}\;\overline{3'6'}\;\overline{4'6'}\;\overline{\phantom{x}'6'} = -5 \cdot \overline{62}\;\overline{31}\;\overline{41}\;\overline{51} = \sigma \frac{1}{y_1 - y_2}, \\
\rho'(y_1' - y_3') &= -5 \cdot \overline{1'3'}\;\overline{2'6'}\;\overline{4'6'}\;\overline{5'6'} = -5 \cdot \overline{63}\;\overline{21}\;\overline{41}\;\overline{51} = \sigma \frac{1}{y_1 - y_3}, \\
\rho'(y_1' - y_4') &= -5 \cdot \overline{1'4'}\;\overline{2'6'}\;\overline{3'6'}\;\overline{5'6'} = -5 \cdot \overline{64}\;\overline{21}\;\overline{31}\;\overline{51} = \sigma \frac{1}{y_1 - y_4}, \\
\rho'(y_1' - y_5') &= -5 \cdot \overline{1'5'}\;\overline{2'6'}\;\overline{3'6'}\;\overline{4'6'} = -5 \cdot \overline{65}\;\overline{21}\;\overline{31}\;\overline{41} = \sigma \frac{1}{y_1 - y_5},
\end{aligned} \tag{3}$$

where

$$\sigma = 25\rho \cdot \overline{12}\;\overline{13}\;\overline{14}\;\overline{15}\;\overline{62}\;\overline{63}\;\overline{64}\;\overline{65}.$$

Thus $T_{(16)}$ is an involutory cubic transformation with the singular tetrahedron $p_2, p_3, p_4, p_5$ and the fixed point $p_1$. Moreover any permutation which affects

$z_6$ gives rise to a cubic transformation with a singular tetrahedron whose vertices are found among the points $p$ and with the same period as the permutation. The equations of the transformation can be obtained in precisely the same way as for $T_{(16)}$.

A point in general position is one of a set of 720 points which form a conjugate set under $C_{6!}$. If a point takes less than 720 positions under $C_{6!}$, it must be unaltered by a certain subgroup $G$ of $C_{6!}$. To such a point there must correspond a class of projective binary sextics, each of which must be self-projective under a binary transformation group $\Gamma$ which is isomorphic with $G$. If all the roots of the sextic are distinct, any permutation of them which can be effected by a binary projectivity defines the projectivity. If however equalities among the roots exist—to fix ideas let $z_1 = z_2$—then to the identical projectivity there corresponds the identical permutation and also the permutation $(z_1, z_2)$. From the formulae (2) we find that the points $y$ which correspond to sextics for which $z_i = z_k (i, k = 1, 2, \cdots, 5)$ lie in the plane $y_i - y_k = 0$; those which correspond to sextics for which $z_i = z_6$ are directions about the point $p_i$. These five points and ten planes are a conjugate system of manifolds under $C_{6!}$, the locus of points for which the invariant $\Delta$ of $S$ vanishes. Since we are interested primarily in the solution of $S$ we shall assume that $\Delta \neq 0$. Then the groups $G$ and $\Gamma$ are simply isomorphic. The types of sextics $S$ invariant under a group $\Gamma$ have been tabulated by Bolza.* From these types the systems of points in question are easily derived.

Because of the projective character of the transition from sextic $S$ to point $y$ it is clear that the locus of points $y$ which map sextics $S$ for which a certain invariant $I$ of $S$ vanishes is a surface, $M = 0$, invariant under $C_{6!}$. Conversely to every surface, $M = 0$, invariant under $C_{6!}$ there corresponds an invariant $I$ of $S$. For if $M = 0$ is invariant under $C_{6!}$, $M$ must be a symmetric function of $y_1, y_2, \cdots, y_5$. Since $\sum y_i = 0$, $M$ is the leading coefficient of a covariant of the quintic, $(t_1 - y_1 t_2) \cdots (t_1 - y_5 t_2)$. Then $M = 0$ expresses that $\infty$ is a certain covariant point of $y_1, \cdots, y_5$, or that $z_6$ is a certain covariant point of $z_1, \cdots, z_5$. The condition that $z_i$ be a similar covariant point of the other $z$'s is that $M = 0$ for all points conjugate to $y$, and this is satisfied, since $M = 0$ is invariant under $C_{6!}$. But this condition implies that the roots $z$ satisfy a rational invariant relation, $I = 0$.

The invariant surfaces therefore are merely the invariants of the sextic

$$6t_2(t_1 - y_1 t_2) \cdots (t_1 - y_5 t_2), \qquad \sum y_i = 0,$$

a sextic in a typical form employed by Clebsch (*Binäre Formen*, p. 351); the invariants of such forms are calculated there up to the sixth order. Another

*Mathematische Annalen, vol. 30 (1887), p. 546; American Journal of Mathematics, vol. 10 (1888), p. 47.

mode of procedure is to express the invariants of $S$ in terms of the differences of the roots $z$. By identifying $z_6$ with $\infty$ and $z_1, z_2, \cdots, z_5$ with $y_1, y_2, \cdots, y_5$ we get the corresponding invariant surface from the coefficient of the highest power of $z_6$. The index of this power is the degree $n$ of $I$ in the coefficients of $S$. The weight $3n$ of $I$ is the order in the differences. But $n$ differences in each term of $I$ contain $z_6$. Thus the order of the corresponding invariant surface $M$ in $y$ is $2n$. Since $M$ must contain all the points to the same multiplicity $k$ and also must be invariant under $T_{(16)}$, we find that $k = n$. Hence

*All surfaces invariant under $C_{6!}$ are rational integral functions of a set of surfaces $M_4$, $M_8$, $M_{12}$, $M_{20}$, $M_{30}$ of orders* 4, 8, 12, 20, 30 *respectively and of multiplicities* 2, 4, 6, 10, 15 *respectively at the points $p$. These surfaces correspond to the members of a complete system of invariants of $S$ of degrees* 2, 4, 6, 10, 15 *respectively.*

The square of $M_{30}$ can be expressed by means of the others; whence the general invariant surface $M$ either is of order $4k$ with $2k$-fold points at $p_i$ or is the aggregate of such a surface and $M_{30}$. The discriminant $\Delta$ of degree ten corresponds to a surface of order twenty, which as we have seen must be the square of the ten planes on $p$. This surface has 12-fold points at $p_i$ instead of the normal 10-fold points. This is accounted for the fact that the $\infty^2$ directions about $p_i$ also map sextics for which $\Delta = 0$. In general if the multiplicity at $p_i$ is greater than the normal multiplicity, $2k$, the ten planes are part of the invariant surface.

The simplest linear system of surfaces which is unaltered by $C_{6!}$ is the system of $\infty^4$ quadrics on the five points $p$. These quadrics map $S_3$ on a cubic surface in $S_4$, whence there is among five linearly independent quadrics on $p$ only a single cubic relation. There are

$$(n+1)(n+2)(n+3)(n+4)/4! - (n-2)(n-1)n(n+1)/4! = \frac{(n+1)(n^2+2n+2)}{2}$$

linearly independent forms of order $n$ in these five quadratic forms. This is exactly the number.

$$\frac{1}{3!}(2n+1)(2n+2)(2n+3) - \frac{5}{3!}n(n+1)(n+2) = \frac{n+1}{2}(n^2+2n+2),$$

of linearly independent forms of order $2n$ with $n$-fold points at $p_i$. Hence every surface invariant under $C_{6!}$ can be expressed as an algebraic form with the five quadrics as variables. For purposes of manipulation however it is convenient to have a symmetrical set of quadrics on the points $p_i$. Since the five points are permuted by $C_{120}$, each member of such a set would be unaltered by a subgroup of $C_{120}$ whose index is the number of members. This number should be five at least and we look for quadrics invariant under a metacyclic subgroup $C_{20}$ of $C_{120}$ whose index is 6. Selecting the particular subgroup generated by $T_{(12345)}$ and $T_{(2354)}$ we find that the quadric required is

$$(y_1-y_2)(y_5-y_3)+(y_2-y_3)(y_1-y_4)+(y_3-y_4)(y_2-y_5)$$
$$+(y_4-y_5)(y_3-y_1)+(y_5-y_1)(y_4-y_2).$$

By using the even permutations the following set of six conjugate quadrics is obtained:

$$\begin{aligned}
2A &= (25)(13)+(51)(42)+(14)(35)+(43)(21)+(32)(54), \quad (ik)=(y_i-y_k),\\
2B &= (12)(53)+(23)(14)+(34)(25)+(45)(31)+(51)(42),\\
2C &= (53)(41)+(34)(25)+(42)(13)+(21)(54)+(15)(32),\\
2D &= (31)(45)+(24)(53)+(25)(41)+(15)(32)+(43)(21),\\
2E &= (31)(24)+(12)(53)+(25)(41)+(54)(32)+(43)(15),\\
2F &= (42)(35)+(23)(14)+(31)(52)+(15)(43)+(54)(21).
\end{aligned} \tag{4}$$

There must exist among these quadrics a linear relation (due to the identities of the type $(12)(34)+(13)(42)+(14)(23)=0$; this is necessarily symmetrical,

$$A+B+C+D+E+F\equiv 0. \tag{5}$$

The even permutation group of the $y$'s, a $C_{60}$, permutes the six quadrics as the icosahedral group permutes the diagonals of the icosahedron; the odd permutations permute the quadrics oddly into the quadrics with changed sign. The Cremona transformation $T_{(16)}$ permutes the quadrics in the order $(AE)(BC)(DF)$ to within a common outstanding factor. Thus to within common factors the quadrics are permuted in all possible ways by the $C_{6!}$. Each quadric is unaltered by one of a second set of six conjugate subgroups of $C_{6!}$ of order 120. The cubic identity satisfied by these quadrics must be unaltered by $C_{6!}$ and therefore must be symmetric in $A, \cdots, F$. Since $\Sigma A=0$, this identity is

$$A^3+B^3+C^3+D^3+E^3+F^3\equiv 0. \tag{6}$$

Every invariant surface can be expressed in terms of these quadrics, and being invariant must be a symmetric or alternating rational integral function of them. Conversely every such function determines a surface invariant under $C_{6!}$. For example, $\Sigma A^5$ does not vanish, since the quadrics satisfy only the relations, $\Sigma A=0$ and $\Sigma A^3=0$. The odd permutations of $C_{120}$ change the sign of $\Sigma A^5$, hence it is an alternating invariant $S$ of degree 5, i. e., to within a numerical factor, $\sqrt{\Delta}$. Also $\Pi(A-B)$ does not vanish and is unaltered by both the even and the odd permutations of $C_{120}$. It must be then the skew invariant of $S$ of degree 15. The symmetric functions, $\Sigma A^2$, $\Sigma A^4$, and $\Pi A$ correspond to invariants of $S$ of degrees 2, 4. and 6 respectively.

To identify the preceding results with known systems let us define, following

Clebsch and Gordan, some invariants and covariants of the sextic. Let

$$
\begin{gathered}
S = a_x^6, \qquad k = k_x^4 = (ab)^4 a_x^2 b_x^2, \\
l = l_x^2 = (ak)^4 a_x^2, \qquad m = m_x^2 = (kl)^2 k_x^2, \qquad n = n_x^2 = (km)^2 k_x^2, \\
A = (ab)^6, \qquad B = (kk_1)^4, \qquad C = (kk_1)^2 (kk_2)^2 (k_1 k_2)^2, \\
D = (ln)^2, \qquad R = (lm)(ln)(mn), \qquad \sqrt{\Delta} = \Pi (z_i - z_k) \qquad (i<k).
\end{gathered} \tag{7}
$$

Adjusting the numerical coefficient in the formula for $\Delta$ given by Gordan (*Invarianten-Theorie*, p. 295) we have

$$
\begin{aligned}
-\frac{1}{2\cdot 3^4}\Delta = {} & 3\cdot 2^7 A^5 - 3\cdot 2^4\cdot 5^3 A^3 B - 2^4\cdot 5^4 A^2 C \\
& + 3\cdot 2\cdot 5^5 (AB^2 + BC) + 3^2\cdot 5^5 D.
\end{aligned} \tag{8}
$$

Let the invariants $A$, $B$, $C$, $D$, and $R$ be calculated for the sextic,

$$6t_2 \prod_1^5 (t_1 - y_i t_2),$$

and expressed symmetrically in terms of the quadrics $A, \cdots, F$. These quadrics satisfy a sextic equation, $\Sigma = 0$, where

$$\Sigma = (Q-A)(Q-B)\cdots(Q-F) = Q^6 + 15q_2 Q^4 + 15q_4 Q^2 + 6q_5 Q + q_6. \tag{9}$$

The coefficients of $\Sigma$ are integral in $A$, $B$, $C$, $\sqrt{\Delta}$, and the square root of the discriminant of $\Sigma$ is a numerical multiple of $R$. The result of the comparison is

$$
\begin{gathered}
15q_2 = \frac{5}{3}A, \qquad 15q_4 = \frac{5}{3}A^2 - \frac{5^3}{2\cdot 3^2}B, \\
q_6 = -\frac{5}{3^3}A^3 + \frac{5^3}{2\cdot 3^3}AB + \frac{5^4}{3^4}C, \qquad 6q_5 = \frac{1}{3^5}\sqrt{\Delta}.
\end{gathered} \tag{10}
$$

Another set of fundamental invariants is defined by the equations,

$$
\begin{gathered}
\bar{A} = \frac{5}{9}A, \qquad \bar{B} = \frac{1}{2\cdot 9^2}(8A^2 - 75B), \\
\bar{C} = \frac{5}{2\cdot 9^3}(-2^3\cdot 13A^3 + 3^2\cdot 5^2 AB + 2\cdot 3^2\cdot 5^3 C),
\end{gathered} \tag{11}
$$

where $\bar{B}$ and $\bar{C}$ both vanish if $S$ has a triple root. Then the sextic $\Sigma$ has the particularly simple form

$$\Sigma = (Q^2 + \bar{A})^3 + 15\bar{B}Q^2 + \frac{1}{3^3}\sqrt{\Delta}\,Q + \bar{C} = 0. \tag{12}$$

If we let

$$\sqrt{\Delta_\Sigma} = \Pi (A-B)\cdots(A-F)(B-C)\cdots(B-F)\cdots(E-F), \tag{13}$$

then

$$\sqrt{\Delta_\Sigma} = -\frac{2^2\cdot 5^{10}}{3^6}R. \tag{14}$$

The form problem of the $C_{6!}$ reads as follows:

(15) *Given the numerical values of* $A$, $B$, $C$, $\Delta$, *invariant under* $C_{6!}$, *to find the ratios of the coördinates of a point* $y$ *for which these quantities take the assigned values.*

The given values determine the numerical values of $B/A^2$, $C/A^3$, and $\Delta/A^5$, whence $y$ is a meeting point of invariant surfaces of orders 8, 12 and 20. Apart from the points $p$ of multiplicities 4, 6, and 10 respectively, these surfaces meet in $8\cdot12\cdot20 - 5\cdot4\cdot6\cdot10 = 720$ points which form a conjugate set under $C_{6!}$. Thus all the solutions of $PC_{6!}$ are obtained rationally from one solution by means of the transformations of $C_{6!}$.

If the numerical value which the product of the ten planes, $\sqrt{\Delta}$, takes for the point $y$ be given it is not possible to determine a set of 360 points conjugate under $C_{6!2}$ by means of three absolute invariants alone. As before, the known ratios $B/A^2$, $C/A^3$, and $\Delta/A^5$ determine *two* sets of 360 conjugate points. If also the value $R$ be given, the known ratios $B/A^2$, $C/A^3$, $R/(\sqrt{\Delta})^3$ determine *three* sets of 360 conjugate points, since the surfaces are of orders 8, 12, and 30 with 4-, 6-, and 15-fold points respectively at $p_i$. But there is an identity (Gordan, *Invarianten-theorie*, pp. 290–1) which expresses the square of $R/(\sqrt{\Delta})^3$ in terms of $B/A^2$, $C/A^3$ and $\Delta/A^5$. Since this identity is cubic in $\Delta/A^5$, only one of the above three sets is contained in the two sets. Thus the form-problem of $PC_{6!2}$ reads:

(16) *Given the numerical values of the invariants* $A$, $B$, $C$, $\sqrt{\Delta}$, *and* $R$ *subject to the identity which connects them, to find the ratios of the coördinates* $y$ *of one of the associated set of* 360 *points conjugate under* $C_{6!2}$.

We have to show finally how the roots $z$ of the given sextic $S$ can be expressed rationally in terms of a solution, $y$, of $PC_{6!2}$. First let us assume that $\Delta \neq 0$ and $R \neq 0$, for otherwise $S$ can be solved by means of radicals. The invariants $A$, $B$, $C$, $\Delta$, and $R$ of $S$ are calculated and a value of $\sqrt{\Delta}$ is adjoined. These values are identified with the numerical values in $PC_{6!2}$. The solution of $PC_{6!2}$ furnishes a sextic

$$S' = 6t_2(t_1 - y_1t_2)\cdots(t_1 - y_5t_2)$$

which is projective to $S$. If we denote the invariants and covariants of $S'$ by primed letters, then

$$(17)\quad I_2^2(lx)^2 = \rho I_2'^2(l't)^2, \qquad I_2(mx)^2 = \rho I_2'(m't)^2, \qquad (nx)^2 = \rho(n't)^2.$$

Here $I_2$ is a non-vanishing invariant of weight 6 which can always be formed from $A$, $B$, $C$, and $\sqrt{\Delta}$ unless $A = B = C = 0$, in which case $S$ is solvable by radicals. $\rho$ is a power of the determinant of the projectivity that transforms $S'$ into $S$. The determinant of the system (17) in the variables $x_1^2$, $2x_1x_2$, and $x_2^2$ is $I_2^3R$ and is not zero. From this system we can find the

values of $x_1^2/\rho$, $2x_1x_2/\rho$, and $x_2^2/\rho$ in terms of the coefficients of $S$, of the coördinates $y_1, \cdots, y_5$ and of $t_1$ and $t_2$. Then from

$$z = \frac{\frac{1}{\rho}x_1^2}{\frac{1}{\rho}x_1x_2} \quad \text{or} \quad z = \frac{\frac{1}{\rho}x_1x_2}{\frac{1}{\rho}x_2^2}$$

we get the six roots $z_1, z_2, \cdots, z_6$ by assigning the values $y_1, y_2, \cdots, y_5, \infty$ to the ratio $t_1/t_2$.

Hence the solution of $S$ and that of $PC_{6!}$ or the solution of $S$ after the adjunction of $\sqrt{\Delta}$ and that of $PC_{6!2}$ are rationally equivalent problems.

## § 2. *The Sextic Resolvent* $\Sigma$ *of* $PC_{6!2}$.

The given quantities in $PC_{6!2}$ as formulated in (16) determine rationally the coefficients of the sextic $\Sigma$ in (9). Also, according to (14), $\sqrt{\Delta_\Sigma}$ is rationally known from $PC_{6!2}$. The Galois group of $\Sigma$ is of order 360 and is simply isomorphic with $C_{6!2}$. Since the roots of $\Sigma$ are rational functions of the required coordinates $y$ in $PC_{6!2}$, $\Sigma$ is in a sense a rational resolvent of $PC_{6!2}$. In order to complete the connection between $\Sigma$ and $PC_{6!2}$ we require formulae which give the ratios of the coordinates $y$ in terms of the roots $A, \cdots, F$ of $\Sigma$. From the formulae (4) we get at once

$$\begin{array}{lll}
\tfrac{1}{2}(A+B) = (51)(42), & \tfrac{1}{2}(B+C) = (34)(25), & \tfrac{1}{2}(C+E) = (13)(42), \\
\tfrac{1}{2}(A+C) = (53)(41), & \tfrac{1}{2}(B+D) = (31)(45), & \tfrac{1}{2}(C+F) = (54)(21), \\
\tfrac{1}{2}(A+D) = (43)(21), & \tfrac{1}{2}(B+E) = (12)(53), & \tfrac{1}{2}(D+E) = (25)(41), \\
\tfrac{1}{2}(A+E) = (32)(54), & \tfrac{1}{2}(B+F) = (23)(14), & \tfrac{1}{2}(D+F) = (24)(53), \\
\tfrac{1}{2}(A+F) = (25)(13), & \tfrac{1}{2}(C+D) = (15)(32), & \tfrac{1}{2}(E+F) = (15)(43).
\end{array} \tag{18}$$

From these pairs of differences the ratios of four linearly independent differences can be derived in a variety of ways. One verifies at once that

$$\begin{aligned}
\rho(12) &= -(A+D)(B+E)(C+F), \\
\rho(13) &= \phantom{-}(A+D)(B+E)(B+D), \\
\rho(14) &= \phantom{-}(A+D)(A+C)(C+F), \\
\rho(15) &= \phantom{-}(E+F)(B+E)(C+F),
\end{aligned} \tag{19}$$

where

$$\rho = 8\cdot(12)^2(34)(35)(45).$$

From these ratios of the differences we find the required ratios of the coördinates $y$,

$$(20)\qquad \begin{aligned} 5y_1 &= (12)+(13)+(14)+(15)\\ 5y_2 &= -4(12)+(13)+(14)+(15)\\ &\cdots\cdots\cdots\cdots\end{aligned}$$

Hence the solution of $\Sigma$ and the solution of $PC_{6!2}$ are rationally equivalent problems.

Since the solutions of $PC_{6!2}$ and the solution of $S$ after the adjunction of $\sqrt{\Delta}$ are also rationally equivalent, it must be possible to exhibit $\Sigma$ directly as a rational resolvent of $S$. This is accomplished by passing from the quadrics $A, \cdots, F$ to functions of the differences $(z_i - z_k)$ by means of formulae (2). Dropping the factor, $25\Pi_{i=1}^{5}(i6)$, from the six we obtain functions, $A', \cdots, F'$, of weight 3 and of order 1 in a particular root. They arise from $A, \cdots, F$ by replacing each term $(ik)(lm)$ by the term $(ik)(lm)(n6)$. These new functions $A', \cdots, F'$ are the roots of a rational resolvent of $S$ after adjunction of $\sqrt{\Delta}$. They have been used by previous writers on this subject, but in these earlier accounts the introduction of the functions seems to be somewhat fortuitous. Here their existence and properties have been shown to be direct consequences of quite elementary geometric group-theory. Moreover the inverse process of expressing the roots of $S$ in terms of the roots of $\Sigma[(19),(20),(17)]$ is as readily suggested and accomplished by the use of familiar formulae.

The functions, $A', \cdots, F'$ of the six roots $z$ seem to have been first used by Joubert,* who has noted their most striking properties and has calculated the equation which they satisfy. A similar study is found in a paper by H. W. Richmond.† Joubert's main interest in this sextic apparently lay in its utility as a resolvent of the quintic.‡

From a quite different point of view, H. Maschke § developed another particular sextic,

$$(21)\qquad \Phi = y^6 - 6F_8y^4 + 4F_{12}y^3 + 9F_8^2y^2 - 12F_{20}y + 4F_{24} = 0.$$

The coefficients $F_i$ are quaternary forms of order $i$ in the variables $z_1, z_2, z_3, z_4$ which are invariant (or which at most change in sign if $i \not\equiv 0 \bmod 8$) under a group of linear substitutions of order 64.720. The group is obtained by effecting transformations of the periods on the Borchardt moduli, $z_i$, for the hyperelliptic thetas of genus 2, where

$$(22)\qquad \begin{aligned} z_1 &= \theta_5 = \vartheta_5(0,0;2\tau_{11},2\tau_{12},2\tau_{22}), & z_3 &= \theta_{23} = \vartheta_{23}(0,0;2\tau_{11},2\tau_{12},2\tau_{22}),\\ z_2 &= \theta_4 = \vartheta_4(0,0;2\tau_{11},2\tau_{12},2\tau_{22}), & z_4 &= \theta_{01} = \vartheta_{01}(0,0;2\tau_{11},2\tau_{12},2\tau_{22}),\end{aligned}$$

* Comptes Rendus (1867), p. 1025 and p. 1080.

† Quarterly Journal of Mathematics, vol. 31 (1899), p. 57.

‡ See the historical account by KLEIN, *Ikosaeder*, II, 1, §§ 4–5, p. 148.

§ Mathematische Annalen, vol. 30 (1887), p. 496; in particular p. 506, (14).

The roots of $\Phi = 0$ are quartic forms in the $z$'s which I shall denote by $\Phi_A$, $\cdots$, $\Phi_F$ [see Maschke, l. c. (7) and (11)], each of which is unaltered by the same group, $\Gamma_{64}$, of linear substitutions or by an isomorphic group, $\Gamma_{16}$, of collineations. They satisfy the two identities

$$\sum \Phi_A \equiv 0, \qquad \sum \Phi_A \Phi_B \Phi_C \Phi_D = \tfrac{1}{4}(\sum \Phi_A \Phi_B)^2. \tag{23}$$

On the other hand, Bolza* has obtained the values of the invariants of the binary sextic in terms of the zero values of the hyperelliptic thetas, $\vartheta_{ik}(0, 0; \tau_{11}, \tau_{12}, \tau_{22})$, associated with the sextic. The squares of the ten even thetas can be expressed as quadratic functions of the $z$'s and therefrom the invariants can be derived as functions of the forms $F_i$ [Bolza, loc. cit., (30), (32); Maschke, loc. cit., (21) and (23)],

$$A^0 = \frac{\rho^2}{3\cdot 5\cdot 2^7}\frac{F_{12}^2 - F_{24}}{F_8 F_{12} - F_{20}}, \qquad B^0 = \frac{\rho^4}{3^4\cdot 5} F_8, \qquad C^0 = -\frac{\rho^6}{3^6\cdot 5} F_{12},\dagger$$
$$\Delta^0 = \frac{2^8\rho^{10}}{3^4}(F_8 F_{12} - F_{20}), \qquad \rho = \frac{(2\pi i)^2}{\omega_{11}\omega_{22} - \omega_{21}\omega_{12}}. \tag{24}$$

Maschke‡ has remarked further that by means of a Tschirnhausen transformation, $y = x^2 + \lambda x + \mu$, the general sextic can be transformed into $\Phi = 0$ by solving a quartic equation. Then the known values of $F_i$ determine the absolute invariants formed from $A^0$, $B^0$, $C^0$, $\Delta^0$. If therefore the periods and the thetas be calculated for a normal hyperelliptic integral associated with a binary sextic which has these absolute invariants, values of $\sqrt{\rho} z_i$ in (24) are obtained and these in turn determine the roots $\Phi_A$ of $\Phi = 0$. Thus the general sextic is solved by means of hyperelliptic modular functions.

Commenting on this paper, F. Brioschi§ notes that the sextic $\Phi$ can be written as

$$(y^3 - 3F_8 y + 2F_{12})^2 + 12(F_8 F_{12} - F_{20})y - 4(F_{12}^2 - F_{24}) = 0. \tag{25}$$

If we use Bolza's invariants and set $\xi = \rho^2 y/3$, this becomes

$$(\xi^3 - 5\cdot 3^3 B^0 \xi - 10\cdot 3^3 C^0)^2 + \frac{\Delta^0}{4^3}(\xi - 5\cdot 2^7 A^0) = 0. \tag{26}$$

If

$$\xi = 5\cdot 2^7 A^0 - t^2,$$
$$T = t^6 - 3\cdot 5\cdot 2^7 A^0 t^4 + 3\cdot 5(5\cdot 4^7 A^{0^2} - 9B^0)t^2$$
$$+ \frac{\sqrt{\Delta}}{8} t - 10(5^2\cdot 4^{10} A^{0^3} - 5\cdot 12^3 A^0 B^0 - 3^3 C^{0^2}) = 0. \tag{27}$$

* Mathematische Annalen, vol. 30 (1887), p. 478.

† The $A^0$, $B^0$, $C^0$, $\Delta^0$, here refer to BOLZA's $A$, $B^2$, $C^3$, $\Delta$.

‡ Rendiconti della R. Accademia dei Lincei, series 4 (1888), p. 181.

§ Rendiconti della R. Accademia dei Lincei, series 4 (1888), p. 183.

In later papers Brioschi* develops more fully the connection between these sextics and the theta moduli.

The sextic $T$ is in Joubert's normal form. By comparing it with $\Sigma$ in (12), the relations between the invariants of Bolza and those of §1 appear. Clearly the solution of $T$ or $\Sigma$ implies the solution of Maschke's sextic $\Phi$ and conversely. We shall find farther on [see (32)] the rational transformation inverse to (27).

The formulæ given above are illuminated by some geometric considerations allied with the papers of H. W. Richmond.† Denote the quadrics $A$ by $A(y^2)$, their polarized forms by $A(yy')$; the quartics $\Phi_A$ by $\Phi_A(z^4)$, their polarized forms by $\Phi_A(z^3z')$. If we polarize the identity (6), $\Sigma A^3(y^2) \equiv 0$, we get the identity, $\Sigma A^2(y^2)A(yy') \equiv 0$. But this expresses that, for a given point $y$, the quadric, $\Sigma A^2(y^2)A(y'^2) = 0$, has a double point at $y$. For given $y'$ we have the condition that a quadric on the six points $p_i$ and $y'$ may have a double point at $y$. Hence

(28) *The equation,* $\Sigma A^2(y^2)A(y'^2) = 0$, *is, for given* $y$, *a quadric cone on the points* $p$ *with vertex at* $y$; *for given* $y'$, *the Weddle quartic surface with double points at the points* $p$ *and* $y'$.

The quartics $\Phi_A$ are unaltered by a collineation $\Gamma_{16}$ determined by a Kummer surface, and every such quartic is a linear combination of the $\Phi_A$'s, say $\Sigma\lambda_A\Phi_A = 0$. Since $\Sigma\Phi_A = 0$ we can suppose that $\Sigma\lambda_A = 0$. The condition that $\Sigma\lambda_A\Phi_A = 0$ be a Kummer surface is then easily shown to be $\Sigma\lambda_A^3 = 0$. Hence $\Sigma A(y'^2)\Phi_A = 0$ is for given point $y'$ a Kummer quartic. It is easily verified‡ that the fundamental sextic of this Kummer quartic and the sextic $y_1', \cdots, y_5', \infty$ are projective; whence

(29) *The Weddle quartic surface,* $\Sigma A(y'^2)A^2(y^2) = 0$, *and the Kummer quartic surface,* $\Sigma A(y'^2)\Phi_A(z^4) = 0$, *are birationally equivalent.*

If on the other hand we polarize the quartic identity (23) we get the new identity

$$\Sigma\Phi_A(z^3z')\Phi_B(z^4)\Phi_C(z^4)\Phi_D(z^4) - \tfrac{1}{2}[\Sigma\Phi_A(z^4)\Phi_B(z^4)][\Sigma\Phi_A(z^3z')\Phi_B(z^4)] \equiv 0.$$

Interchanging $z$ and $z'$ we have the condition that the quartic surface in the variables $z$,

$$\Sigma\Phi_A(z^4)\Phi_B(z'^4)\Phi_C(z'^4)\Phi_D(z'^4) - \tfrac{1}{2}[\Sigma\Phi_A(z'^4)\Phi_B(z'^4)][\Sigma\Phi_A(z^4)\Phi_B(z'^4)] = 0, \tag{30}$$

may have a double point at $z'$. Being unaltered by $\Gamma_{16}$ it must have 16 double points, whence

*Acta Mathematica, vol. 12 (1889), p. 83; Annales de l'Ecole Normale Supérieure, (3), vol. 12 (1895), p. 343.

†Quarterly Journal of Mathematics, vol. 31 (1899), p. 125; vol. 34 (1902), p. 117.

‡The necessary formulæ are given by HUDSON, *Kummer's Quartic Surface*, pp. 81–2.

(31) *The quartic surface* (30) *is a Kummer quartic surface with double points at the* 16 *points conjugate to* $z'$ *under* $\Gamma_{16}$.

The coefficients of $\Phi_A(z^4), \cdots,$ in (30) after being modified so that their sum is zero can be identified with the coefficients of $\Phi_A(z^4)$ in (29). Expressing each coefficient in terms of the symmetric functions of the $\Phi_A$'s, i. e., the $F_i$, and of a particular $\Phi_A$ we find with little trouble that

$$A = -\Phi_A^3 + 3F_8\Phi_A - 2F_{12}. \tag{32}$$

This formula furnishes the Tschirnhausen transformation of the sextic $\Phi$ into the sextic $\Sigma$.

If we interpret the $A$'s as homogeneous point coordinates, and the $\Phi_A$'s as space coordinates in $S_4$, then $\Sigma A^3 = 0$ is the equation of a diagonal cubic surface, $K_3$, whose reciprocal is a quartic envelope, $R_4$. The equation of $R_4$ is precisely the quartic identity (23). Thus we have the particularly simple interpretation of (27) and (32):

(33) *The inverse Tschirnhausen transformations* (27) *and* (32) *from the Joubert to the Maschke normal form* (*and vice versa*) *represent the process of passing from point to tangent space* (*and vice versa*) *of the reciprocal loci* $K_3$ *and* $R_4$ *in the space* $S_4$.

If the Maschke sextic be solved as already indicated, then the roots of $\Sigma$ are obtained from (32). Consequently also the solution of $PC_{6!}$ in terms of hyperelliptic modular functions of genus two is known. Ultimately then the roots of the given sextic $S$ are expressed in terms of such functions, i. e. in terms of transcendental functions of three variables, the moduli $\tau_{11}, \tau_{12}, \tau_{22}$. This however can hardly be accepted as a final solution. The $PG_{360}$ contains only two independent variables and by the use of it in place of the modular functions an essential simplification is obtained.

## § 3. *The Reduction of* $PC_{6!2}$ *to the* VALENTINER *Form-Problem.*

The even permutations which do not alter $z_6$ give rise to a collineation subgroup $C_{60}$ of $C_{6!2}$. This subgroup is one of a set of six conjugate subgroups, $G_{60}^{(1)}, \cdots, G_{60}^{(6)}$, the root $z_i$ being associated with $G_{60}^{(i)}$. The quadrics $A, \cdots, F$ also are permuted in all possible even ways by $C_{6!2}$. Therefore $C_{6!2}$ contains a second set of six conjugate subgroups, $G_{60}^{(A)}, \cdots, G_{60}^{(F)}$, the quadric $A$ being unaltered by $G_{60}^{(A)}$, etc. The same set of subgroups $G_{60}^{(A)}$ is defined by the functions $A', \cdots, F'$ of the roots $z$. If then we begin with the quadrics $A, \cdots, F$ and form from them six functions $A'', \cdots, F''$ in the same way that the $A'$s are formed from $z_i$, these new functions will be unaltered respectively by a set of conjugate subgroups $G_{60}$ which are different from $G_{60}^{(A)}, \cdots, G_{60}^{(F)}$. The groups $G_{60}^{(A'')}, \cdots, G_{60}^{(F'')}$ must be identical to within their order with the groups $G_{60}^{(1)}, \cdots, G_{60}^{(6)}$, since $C_{6!2}$ is known to contain only two

systems of conjugate icosahedral subgroups. Let us then name these functions $H^{(1)}, \cdots, H^{(6)}$. Each $H$ is of order 3 in the differences of $A, \cdots, F$, of order 6 in the differences of the $y$'s and of maximum order 3 in a particular $y$. Also since $H_6$ is unaltered by $G_{60}^{(6)}$ or $C_{60}$ it must be a seminvariant of the quintic with roots $y$ of degree 3 and weight 6. The quintic has only one such covariant—its canonizant of order 3. Thus $H_6 = 0$ must be the condition that $\infty$ be a canonizant point of $y_1, y_2, \cdots, y_5$. Similarly $H_5 = 0$ is the condition that $y_5$ be a canonizant point of $y_1, \cdots, y_4, \infty$; etc. The surfaces, $H_i = 0$, are sextic surfaces with triple points at $p_i$ which satisfy relations similar to (5) and (6). Adjusting the numerical factor so that

$$\sum_{i=1}^{6} H_i \equiv 0, \qquad \sum_{i=1}^{6} H_i^3 \equiv 0,$$

we find that

$$H_i = y_i^6 - 3y_i^4 s_2^{(i)} + 2y_i^3 s_3^{(i)} - y_i^2 [(s_2^{(i)})^2 - 70 s_4^{(i)}] + 8 y_i s_2^{(i)} s_3^{(i)}$$

$$+ (s_2^{(i)})^3 + 8(s_3^{(i)})^2 - 20 s_2^{(i)} s_4^{(i)} \quad (i = 1, 2, 3, 4, 5); \tag{36}$$

$$H_6 = -s_2^3 - 10 s_3^2 + 20 s_2 s_4,$$

where

$$s_k = \Sigma y_1 y_2 \cdots y_k, \qquad s_k^{(i)} = s_k - y_i s_{k-1}^{(i)}, \qquad s_0^{(i)} = 1.$$

The Valentiner group of 360 ternary collineations, $G_{360}$, is isomorphic with $C_{6!2}$ and has also two sets of conjugate icosahedral subgroups, $G_{60}^{(1)}, \cdots, G_{60}^{(6)}$; and $G_{60}^{(A)}, \cdots, G_{60}^{(F)}$. As invariants under the subgroups we have two sets of six conics, $(k_1 x)^2, \cdots, (k_6 x)^2$; and $(ax)^2, \cdots, (fx)^2$ [See $C2$, § 1]. If each conic be multiplied by its reciprocal, the two sets of six products are permuted by $G_{360}$ without extraneous factors. Hence the forms

$$\sum_{1}^{6} (k_1 x)^2 \cdot (k_1 k_1' u)^2 \cdot H_1$$

and

$$\sum_{a}^{f} (ax)^2 \cdot (aa' u)^2 \cdot A \tag{37}$$

are *Kleinian forms*, i. e., algebraic forms in the three sets of variables $x_1, x_2, x_3$; $u_1, u_2, u_3$; $y_1, y_2, \cdots, y_5$, which are merely multiplied by a factor when the variables $x$ and $u$ are contragrediently transformed by the collineations of $G_{360}$ and simultaneously the variables $y$ are transformed by the corresponding substitutions of $C_{6!2}$. The correspondence between $G_{360}$ and $C_{6!2}$ is established by associating $H_i$ with $(k_i x)^2$. From these two forms we derive the covariant connex (1, 1)

$$\sum_{1}^{6} \sum_{a}^{f} H_1 \cdot A \cdot (k_1 k_1' a)^2 \cdot (k_1 aa')(k_1 x)(aa' u) \tag{38}$$

which is also a Kleinian form. This connex viewed as a collineation has three fixed points which can be separated by means of an "accessory" cubic equation. One of these fixed points is a so-called "covariant point" by means of which the solution of $PC_{6!2}$ is reduced to $PG_{360}$.

Simpler algebraically than this direct transition from $PC_{6!2}$ to $PG_{360}$ is the solution of the resolvent $\Sigma$ in terms of $PG_{360}$. This is accomplished exactly as outlined for the general sextic in $C2$ (in particular §4), the expressions being considerably shorter owing to the fact that in the case of $\Sigma$, $a_1 = a_3 = 0$. For example, the accessory cubic equation [$C2$, §4, (1), (2), (3), (4)] is

$$\sigma^3 + J\sigma - K = 0,$$

where

(39)
$$J = \tfrac{1}{2}[15q_2^2q_4 - \tfrac{1}{5}q_2q_6 - 2q_4^2],$$
$$K = \tfrac{1}{2}[-45q_2^2q_4^2 + \tfrac{6}{5}q_2q_4q_6 + 8q_4^3 - \tfrac{1}{2}\tfrac{1}{5}q_6^2].$$

From the preceding considerations I conclude that *as a matter of practical convenience* it seems advisable to effect the solution of the diagonal sextic resolvent $\Sigma$ in terms of $PG_{360}$ rather than the given sextic $S$. As the natural bond between $S$ and $\Sigma$, the Cremona $C_{6!2}$ must be introduced.

BALTIMORE, *Oct.* 1, 1910.

# ON THE USE OF THE CO-SETS OF A GROUP*

BY

G. A. MILLER

## §1. *Introduction.*

ABBATI proved in a letter addressed to RUFFINI and dated September 30, 1802, that it is possible to divide all the operators of a group $G$ into sets with respect to an arbitrary subgroup $H$ so that no two of the sets have a common operator. That is, every subgroup of $G$ may be regarded as a modulus with respect to either right or left multiplication. For any subgroup $H$ a set of operators $S_2, S_3, \cdots, S_\rho$ may be selected so that this division can be effected in each of the following four ways:†

$$\begin{aligned} G &= H + HS_2 + HS_3 + \cdots + HS_\rho, \\ &= H + S_2H + S_3H + \cdots + S_\rho H, \\ &= H + S_2^{-1}H + S_3^{-1}H + \cdots + S_\rho^{-1}H, \\ &= H + HS_2^{-1} + HS_3^{-1} + \cdots + HS_\rho^{-1}. \end{aligned}$$

The sets $HS_\alpha$, $S_\alpha H$ $(\alpha = 2, 3, \cdots, \rho)$ have been called Nebengruppen ‡ or *co-sets* of $G$ as regards $H$, the former being called the right co-sets and the latter the left co-sets. When $H$ is added to these co-sets they are called the augmented right co-sets and the augmented left co-sets, respectively.

It has been observed that the operators of each right co-set are evenly distributed among a certain number of the left co-sets, or among all of them, and that the operators of each left co-set are distributed in the same way among the right co-sets. When a right co-set involves an operator which transforms $H$ into itself all of its operators have this property and this right co-set involves exactly the same operators as some left co-set, and vice versa. A necessary and sufficient condition that $H$ gives rise to a multiply transitive substitution group $K$, when $G$ or one of its quotient groups is represented transitively as regards $H$ in the usual manner,§ is that the operators of each right co-set of $G$ as regards $H$ are distributed among all the left co-sets as regards $H$, $\rho$ being greater than 2.‖ The same condition is expressed by saying that whenever $S$

* Presented to the Society (Chicago) December 28, 1910.

† Quarterly Journal of Mathematics, vol. 41 (1910), p. 384.

‡ WEBER, *Lehrbuch der Algebra*, 2d edition, 1899, vol. 2, p. 8.

§ DYCK, Mathematische Annalen, vol. 22 (1883), p. 91.

‖ Proceedings of the American Philosophical Society, vol. 49 (1910), p. 307.

is any operator of $G$ which is not also in $H$ every operator of $G$ is included in the two sets $H$ and $HSH$, and that $\rho$ exceeds 2.

GALOIS called attention to the importance of the special case when each right co-set of $G$ as regards $H$ is identical with some left co-set. That this case expresses the necessary and sufficient condition that $H$ is invariant under $G$ follows from the more general theorem noted above that a necessary and sufficient condition that $H$ is transformed into itself by the operators of a right co-set is that this co-set is identical with some left co-set, and vice versa. This theorem is, in turn, included in the still more general statement that a necessary and sufficient condition that the operators of a co-set transform $H$ into a group having exactly $\gamma$ operators in common with $H$ is that this co-set has exactly $\gamma$ operators in common with some other co-set with respect to $H$. This statement may again be regarded as a special case of the more general theorem which affirms that the total number of operators in $G$ which transform $H$ into a group having exactly $\rho$ operators in common with some other subgroup $K$ is a multiple of the product of the orders of $H$ and $K$ divided by $\rho$.*

When $H_1$ and $H_2$ are any two groups the product $H_1H_2$ represents the totality of distinct operators obtained by multiplying on the left each operator of $H_2$ by each operator of $H_1$. The number of distinct operators in this totality is evidently the product of the orders of $H_1$ and $H_2$ divided by the number of their common operators. A necessary and sufficient condition that this totality constitutes a group is that $H_1H_2 = H_2H_1$. This is a special case of a more general theorem which may be stated as follows: A necessary and sufficient condition that the continued product of the $\lambda$ groups $H_1, H_2, \cdots, H_\lambda$ is a group is that this product is not affected by the cyclic permutations of its factors. That is, $H_1H_2 \cdots H_\lambda$ is a group whenever

$$H_1H_2 \cdots H_\lambda = H_2 \cdots H_\lambda H_1 = \cdots = H_\lambda H_1 H_2 \cdots,$$

as may easily be verified.

The given method of arranging the operators of $G$ in distinct sets, called augmented co-sets, can easily be generalized by observing that the co-sets are completely determined by $H$ and do not depend upon the choice of the operators $S_2, S_3, \cdots, S_\rho$. Hence it follows directly that the *double co-set* $HS_\alpha H_1$, $H_1$ being an arbitrary subgroup of $G$, remains unchanged when $S_\alpha$ is replaced by any other operator of this double co-set. That is, two double co-sets $HS_\alpha H_1$, $HS_\beta H_1$ either have all their operators in common or they have no operator in common.† Each operator of $G$ may therefore be uniquely represented by such double co-sets and $HH_1$; that is, the group may be represented by a system of augmented double co-sets, as follows:

* Bulletin of the American Mathematical Society, vol. 16 (1910), p. 510.

† FROBENIUS, Journal für reine und angewandte Mathematik, vol. 101 (1887), p. 274.

$$G = HH_1 + HS_2H_1 + \cdots + HS_\lambda H_1.$$

When $H_1 = H$ and $G$ or one of its quotient groups is represented as a transitive substitution group with respect to $H$ in the ordinary way, the number of transitive constituents in the subgroup composed of all the substitutions which omit one letter of this substitution group is $\lambda$, if the omitted letter or letters are counted as constituents; and the numbers of distinct operators in these various double co-sets are equal to the products of the order of $H$ and the degrees of the various transitive constituents of this subgroup.* When $H_1 = 1$ these double co-sets reduce to right co-sets and when $H = 1$ they reduce to left co-sets.

The general properties of double co-sets were first studied by FROBENIUS in an article entitled *Ueber die Congruenz nach einem aus zwei endlichen Gruppen gebildeten Doppelmodul*, Journal für die reine und angewandte Mathematik, volume 101 (1887), page 273. Since then the concept of equivalence of operators as regards a double modulus has been employed by various writers. The object of the present article is to extend the abstract theory as regards co-sets with a view to a greater usefulness of this concept. This article is practically a continuation of the one entitled, *Some relations between substitution group properties and abstract groups*, Proceedings of the American Philosophical Society, volume 49 (1910), page 307. The theorem of § 3 which exhibits a dual relation between the degrees of the transitive constituents of various subgroups when the group is represented with respect to these subgroups is perhaps the most useful result of the present paper.

## § 2. *Double co-sets with respect to a single subgroup.*

The double co-set $HS_\alpha H$ involves all the right co-sets which are conjugate under $H$ with $HS_\alpha$. It also includes all the conjugates under $H$ of the left co-set $S_\alpha H$. In fact, each of these two complete sets of conjugates involves exactly the same operators, and is identical with $HS_\alpha H$. The number of distinct conjugates in each of these two sets may be obtained by dividing the order of $H$ by the number of operators common to $H$ and $S_\alpha^{-1}HS_\alpha$ since this is also the number of operators common to the two co-sets $HS_\alpha$ and $S_\alpha H$, and hence it represents the number of left co-sets among which all the operators of $HS_\alpha$ are evenly distributed as well as the number of right co-sets among which the operators of $S_\alpha H$ are evenly distributed. We shall prove in the next paragraph that this same number is also the degree of a transitive constituent of the subgroup composed of all the substitutions omitting one letter in the transitive substitution group $K$ which corresponds to the permutations of the rows when all the operators of $G$ are arranged in the ordinary rectangular form and the operators of $H$ constitute the first row.†

* Proceedings of the American Philosophical Society, vol. 49 (1910), p. 311.

† DYCK, Mathematische Annalen, vol. 22 (1883), p. 91.

To prove this theorem we shall at first assume that $K$ is simply isomorphic with $G$. This is equivalent to assuming that $H$ contains no invariant subgroup of $G$ besides the identity, or that $K$ represents $G$. Let $K_1$ represent the subgroup of $K$ which corresponds to $H$ in this simple isomorphism. Hence $K_1$ is composed of all the substitutions of $K$ which omit a given letter ($a$) and $HS_a$ corresponds to all the substitutions of $K$ which replace $a$ by a given other letter. The conjugates of $HS_a$ are then all the operators of $G$ which correspond to those substitutions of $K$ which replace $a$ by the different letters in a transitive constituent of $K_1$. That is, *if the operators of $G$ are arranged into double co-sets with respect to a single subgroup in the following manner:*

$$G = H + HS_2H + HS_3H + \cdots + HS_\lambda H,$$

*and if $H$ does not involve any invariant subgroup of $G$ besides the identity, so that $G$ may be represented as a transitive substitution group $K$ with respect to $H$, then the operators in each of these double co-sets correspond to all the substitutions of $K$ which replace the letter omitted in the subgroup corresponding to $H$ by the letters in a transitive constituent of this subgroup.*

When $H$ includes an invariant subgroup of $G$ or is itself invariant under $G$, it gives rise to the transitive representation of a quotient group of $G$ and the preceding theorem applies to this quotient group. Hence $\lambda$ represents the number of transitive constituents (each omitted letter being also regarded as a transitive constituent) in the subgroup composed of all the substitutions of this quotient group which omit a given letter, and we may establish a $(1, 1)$ correspondence between the double co-sets and the transitive constituents of this subgroup in such a way that the number of distinct operators in each double co-set is equal to the product of the order of $H$ into the degree of the corresponding transitive constituent. In particular when $H$ is invariant this quotient group is regular and the number of these double co-sets is equal to the degree of this quotient group, as each of the transitive constituents of the given subgroup is of degree one in this special case. Hence we have as a corollary of the given theorem the known theorem that necessary and sufficient conditions that $H$ gives rise to a multiply transitive group are that $H$ contains less than half the operators of $G$ and that $\lambda = 2$.

A necessary and sufficient condition that the transitive substitution group $K$ to which $H$ gives rise is primitive is that $G$ is generated by each double co-set as regards $H$, since this condition is equivalent to the condition that $H$ is a maximal subgroup. A necessary and sufficient condition that the subgroup of $K$ composed of all its substitutions which omit one letter omits exactly $\alpha$ letters is that there are $\alpha - 1$ double co-sets as to $H$ each involving the same number of distinct operators as $H$ does. This results directly from the more general theorem that all the operators of such a double co-set transform $H$ into

groups all having the same number of operators in common with $H$ and that the number of the distinct operators in this co-set is equal to the square of the order of $H$ divided by the number of these common operators. This is, in turn, a special case of a theorem which will be proved in the following section.

### § 3. *Double co-sets with respect to two distinct subgroups.*

A special case of a double co-set with respect to two distinct subgroups was developed by Cauchy to prove the fundamental theorem that every group whose order is divisible by a given prime number involves a subgroup whose order is this prime.* This theorem was announced by Galois, who however did not give any proof in his published papers. Frobenius extended, in the article cited at the close of § 1, the development of the theory of double co-sets and employed this theory to re-establish Sylow's theorem as well as to prove other fundamental related theorems. Recently the present writer pointed out that Sylow's theorem may be proved directly by a slight extension of Cauchy's theorem relating to double co-sets and then following the steps which Cauchy took to establish the special but fundamental case noted above.† These facts may suffice to indicate that the concept of double co-sets relates to fundamental matters in group theory and seems to deserve more attention than it has recently received.

If $H_1$ and $H_2$ are any two subgroups of $G$, all the operators of $G$ may be represented uniquely in either of the following two forms in accord with the results stated above:

$$G = H_1 H_2 + H_1 S_2 H_2 + \cdots + H_1 S_\lambda H_2,$$
$$= H_2 H_1 + H_2 S_2^{-1} H_1 + \cdots + H_2 S_\lambda^{-1} H_1.$$

The second augmented system of double co-sets is evidently composed of the inverses of the operators in first system. To obtain a concrete illustration of some of the properties of these augmented double co-sets it may be observed that when $G$ can be represented as a transitive substitution group $K$ with respect to $H_1$, the substitutions corresponding to $H_1 H_2$ will be composed of all those of $K$ which replace the letter $a$ omitted in all the substitutions of the subgroup $K_1$ corresponding to $H_1$ by the letters of the transitive constituent to which $a$ belongs in the subgroup $K_2$ corresponding to $H_2$. In a similar manner we may observe that all the operators of the double co-sets $H_1 S_\alpha H_2$ ($\alpha = 2, 3, \cdots, \lambda$) correspond to all the substitutions which replace the letter by which $a$ is followed in $S_\alpha$ by all the letters of a transitive constituent in $K_2$. The number of the distinct operators in the double co-set $H_1 S_\alpha H_2$ is therefore equal to the order of

* CAUCHY, *Oeuvres complètes*, 1st series, vol. 9 (1896), p. 358.

† Bulletin of the American Mathematical Society, vol. 16 (1910), p. 510.

$H_1$ multiplied by the degree of the corresponding transitive constituent of $K_2$. In particular, this number is the order of $H_1$ whenever the letter which follows $a$ in $S_a$ does not occur in $K_2$. As a special case we have also the theorem that a necessary and sufficient condition that $H_1 H_2 = G$ is that $K_2$ be transitive.

From the preceding results we may deduce a general theorem as regards the transitive substitution groups which are simply isomorphic with $G$. This theorem may be stated as follows: *If $G$ is represented as a transitive substitution group $K$ and if $K_1$ is the subgroup composed of all the substitutions of $K$ which omit a given letter, then any subgroup $K'$ of $K$ has the same number of transitive constituents as the subgroup which corresponds to $K_1$ has when $K$ is represented as a transitive group with respect to $K'$. Moreover, the transitive constituents in the two given subgroups have the same relative degrees in the two different representations of $G$.*

The significance of this theorem may be inferred from the following illustrations. When the icosahedral group is represented as the alternating group of degree 5, $K_1$ is the alternating group of degree 4 and all the subgroups of order 12 in $K$ are conjugate. If we let $K'$ represent one of the subgroups of order 4 in $K$ it is evident that $K'$ has two transitive constituents of degrees 4 and 1 respectively. Hence when $K$ is represented as a transitive group of degree 15 with respect to $K'$ its subgroups of order 12 will have two transitive constituents of degrees 12 and 3 respectively. On the other hand, if we let $K'$ represent a subgroup of order 6 in $K$, it is clear that it has two transitive constituents of degrees 3 and 2 respectively. Hence when $K$ is represented as a transitive group of degree 10 its subgroups of order 12 will have two transitive constituents of degrees 6 and 4 respectively. Finally, if we let $K'$ be a subgroup whose order is either 5 or 10, it is transitive, and hence the subgroups of order 12 in the icosahedral group are transitive when this group is represented transitively on either 6 or 12 letters.

It should be observed that the given theorem remains true even when $K$ does not represent $G$ but represents merely a quotient group of $G$; that is when $H_1$ either involves an invariant subgroup of $G$ or is itself invariant under $G$. In these cases it is evidently necessary to consider the groups which correspond to the subgroups $K_1$ and $K'$ in the transitive representation instead of these subgroups themselves. The given development exhibits not only an interesting dual relation between the forms of the various subgroups when a group is represented transitively in the various possible ways, but it also exhibits a dual relation between certain abstract group properties and substitution groups, since not only does $\lambda$ express the number of transitive constituents in the group corresponding to $H_2$ when $G$ is represented transitively as regards $H_1$, but the number of distinct operators in each of the double co-sets is equal to the product of the order of $H_1$ and the degree of the corresponding transitive constituent.

The number of distinct operators in each one of the augmented double co-sets is a multiple of the orders of the two subgroups $H_1$, $H_2$. That is, *if $K'$ is any subgroup of any transitive group whatever of order $g$ and of degree $n$, and if $n_1$ is the degree of a transitive constituent of $K'$, then the order of $K'$ is a divisor of $n_1 g/n$.* This limits the orders of the intransitive subgroups of any transitive substitution group and it reduces to Lagrange's theorem when the subgroups are transitive. It may therefore be regarded as a generalization of Lagrange's theorem. It should be observed that this theorem may also be proved directly by employing the fact that the order of the subgroup composed of all the substitutions of $K'$ which omit a given letter is a divisor of $g/n$.

To illustrate the usefulness of the theorem in the last paragraph we may again make use of transitive representations of the icosahedral group. When this group is represented as the alternating group of degree 5, $K'$ can have a transitive constituent of degree 2 only when the order of $K'$ is a divisor of 24, and it can have a transitive constituent of degree 3 only when this order is a divisor of 36, in accord with the given theorem. The only non-cyclic subgroups which have constituents of degree 2 or 3 are those of order 6, and as 6 is a divisor of both the numbers 24 and 36 this agrees with the theorem. On the other hand, when the icosahedral group is represented as a transitive group of degree 6, $K'$ can have a transitive constituent of degree 2 only when the order of $K'$ is a divisor of 20. As a subgroup whose order is divisible by 5 in a transitive group of degree 6 could clearly not have a transitive constituent of degree 2, it results that the order of $K'$ is a divisor of 4 whenever it has a transitive constituent of degree 2. In fact, it is evident that the subgroups of orders 2 and 4 are the only ones which have transitive constituents of degree 2 in the present case. Similarly we observe that the order of $K'$ is a divisor of 6 whenever it has a transitive constituent of degree 3, and that in the present case the subgroups of order 6 have two constituents of this degree. These illustrations may suffice to exhibit how the theorem under consideration may be employed to gain an insight into the structure of transitive substitution groups.

It is known that a necessary and sufficient condition that the product $H_1 H_2$ constitutes a group is that it is equal to its inverse $H_2 H_1$. We proceed to inquire what is implied by the condition that each double co-set is identical with the inverse of a double co-set as regards the same subgroups. That is, we suppose that the two series of augmented double co-sets

$$\begin{aligned} G &= H_1 H_2 + H_1 S_2 H_2 + \cdots + H_1 S_\lambda H_2 \\ &= H_2 H_1 + H_2 S_2^{-1} H_1 + \cdots + H_2 S_\lambda^{-1} H_1 \end{aligned}$$

are identical if the order is not considered. Since each of the two products $H_1 H_2$ and $H_2 H_1$ involves the identity it results that these must be identical when the given condition is satisfied, and hence a first result is that $H_1 H_2$ must be a sub-

group of $G$. The assumption that $H_1 S_a H_2 = H_2 S_a H_1$ clearly implies that $H_1 H_2$ is invariant under $S_a$. That is, *whenever a double co-set as regards two permutable subgroups coincides with the inverse of a double co-set as regards the same two subgroups then the product of these subgroups is invariant under each of these double co-sets.* In particular, a necessary and sufficient condition that $H_1 H_2$ is an invariant subgroup under $G$ is that each double co-set as regards $H_1$, $H_2$ is identical with the inverse of some double co-set as regards these subgroups. If either $H_1$ or $H_2$ is assumed to be the identity we obtain as a special case the theorem noted in the third paragraph of the Introduction.

When $G$ or a quotient group of $G$ is represented as a substitution group as regards $H_1$, the preceding results may also be employed to establish a useful criterion to determine when $H_1 H_2$ constitutes a subgroup of $G$. In fact, since $H_1 H_2$ is composed of all the operators of $G$ corresponding to the substitutions which replace the letter $(a)$ omitted in the subgroup corresponding to $H_1$ in this substitution group by the letters of a transitive constituent which involves $a$ in the subgroup corresponding to $H_2$, it results that *a necessary and sufficient condition that $H_1 H_2$ is a group is that the transitive constituent which involves $a$ in the subgroup corresponding to $H_2$ involves all the letters of one or more transitive constituents in the group corresponding to $H_1$ when $G$, or one of its quotient groups, is represented transitively as regards $H_1$ and $a$ is the letter omitted in the group corresponding to $H_1$.*

The preceding theorem is a special case of another theorem which we proceed to establish. Since $H_1 H_2$ is composed of all the operators of $G$ which correspond to the substitutions, in the transitive group to which $H_1$ gives rise, which replace the letter $a$ omitted in all the substitutions corresponding to $H_1$ by all the letters of the transitive constituent involving $a$ in the subgroup corresponding to $H_2$, the number of operators common to $H_1 H_2$ and $H_2 H_1$ can be directly determined from this substitution group. In fact, a necessary and sufficient condition that $H_1 H_2$ and $H_2 H_1$ have in common all the operators which correspond to the total number of substitutions which replace $a$ by the letter $c$ is that all the letters in the constituent to which $c$ belongs in the subgroup corresponding to $H_1$ are found in the transitive constituent to which $a$ and $c$ belong in the subgroup corresponding to $H_2$. In particular, if all the letters in the latter constituent constitute the letters of two or more constituents in the group corresponding to $H_1$ it results that $H_1 H_2$ and $H_2 H_1$ are identical, as was proved in the preceding paragraph. This proves also that $H_1 H_2$ *cannot involve the inverses of all the operators of a double co-set with respect to* $H_1$, $H_2$.

The co-sets which contain the inverse of at least one of the operators in $H_1 H_2$ can readily be determined when $G$ or a quotient group of $G$ is represented with respect to $H_1$. In fact, if a letter in the transitive constituent involving $a$ in the subgroup corresponding to $H_2$ is replaced, in the subgroup

corresponding to $H_1$, by a letter of another transitive constituent of the former of these two subgroups the double co-set corresponding to the latter constituent involves the inverse of an operator of $H_1 H_2$ and vice versa. To determine the double co-sets which involve the inverse of at least one of the operators of $H_1 H_2$ it is therefore only necessary to observe which of the transitive constituents of the subgroup corresponding to $H_2$ involves at least one letter that succeeds in the subgroup corresponding to $H_1$ a letter of the constituent involving $a$ in the former subgroup. When a set of constituents, including the one involving $a$, of this subgroup involves all the letters of a set of constituents of the group corresponding to $H_1$, the letters of these constituents clearly represent a system of imprimitivity of the substitution group with respect to $H_1$. In particular this group is always imprimitive when $H_1 H_2 \equiv H_2 H_1 < G$.

When $G$ is written as the sum of the subgroup $H$ and a series of corresponding co-sets, as follows:

$$G = H + HS_2 + HS_3 + \cdots + HS_\rho,$$

the series of operators $S_2, S_3, \cdots, S_\rho$ may be so selected that the same totality of co-sets is obtained by replacing each of these operators by its inverse, as was observed in the opening paragraph. This can evidently also be done as regards double co-sets with respect to a single subgroup, but it cannot always be done as regards double co-sets with respect to two distinct subgroups. The truth of this statement may be verified if we let $G$ be the transitive substitution group of order 32 and of degree 8 generated by the substitution $aebf \cdot cgdh$ and the two subgroups

$$H_1 = 1,\ eg \cdot fh,\ bd \cdot eg,\ bd \cdot fh, \qquad H_2 = 1,\ ac \cdot eg.$$

Hence it results that $G$ must have special properties when it is possible to select the operators $S_2, \cdots, S_\lambda$ so that both of the following arrangements are possible when every operator of $G$ is uniquely represented by the former:

$$G = H_1 H_2 + H_1 S_2 H_2 + \cdots + H_1 S_\lambda H_2,$$
$$= H_1 H_2 + H_1 S_2^{-1} H_2 + \cdots + H_1 S_\lambda^{-1} H_2.$$

From the theorem that $H_1 H_2$ cannot involve the inverses of all the operators of a double co-set with respect to $H_1 H_2$ it results that $G$ may always be represented in these two ways if $S_2, \cdots, S_\lambda$ are properly chosen and $\lambda < 4$. It is also clear that this is always possible when $H_1$ gives rise to a doubly transitive substitution group, since all the double co-sets involve operators of order 2 in this special case.

# THE SOUTHERLY DEVIATION OF FALLING BODIES*

BY

WILLIAM H. ROEVER

## INTRODUCTION

In the work of previous writers on the subject of this paper the potential function used for that of the earth's gravitational field of force has, in general, not been more than a first approximation, i. e., a development, in the neighborhood of the initial point of the falling body, which includes terms of only the first order in the independent variables. The circle of reference (parallel of latitude) used has also, in general, been different from that used in experiments for the determination of the same quantity.

From equations (1) and (2) which follow it appears that the southerly deviation is proportional to the square of the height $h$ through which the body falls (at least for sufficiently small values of $h$) and that the constant of proportionality involves the first *and also the second derivatives of the potential function* $f_1$.† Hence the truth of the following statement:

I. The potential function to be used for that of the earth's gravitational field of force should be *a second approximation*, i. e., a development in the neighborhood of the initial point of the falling body, which includes terms of at least the second order in the independent variables.

In experiments for the determination of the southerly deviation of falling bodies, a plumb-line $P_0R$ is supported at the point $P_0$ (Fig. 5), from which the body falls, and the deviation is measured on the level (equipotential) surface which passes through the plumb-bob $R$, from the circle of latitude which passes through $R$. The direction of gravity at $R$ is $RP_0$, and that at $P_0$ is the limiting direction which is approached by the line $RP_0$ as the length $P_0R$ approaches zero. Let us denote by $t$ the limiting position of $P_0R$. It is easy to prove the following statement:

II. In a field of force in which the lines of force are not rectilinear, plumb-

* Presented to the Society (Southwestern Section), November 27, 1909, and November 26, 1910. See also footnote to formula (3).

† In this paper, as in the work of previous writers, a distribution of revolution is assumed, i. e., the earth's field of force is assumed to be the same in every plane through the axis of rotation.

lines* of different lengths which are supported at the same point (in particular a plumb-line $P_0R$ and the line $t$) do not coincide.

In order to be able to compare the results of experiment with those of theory, the same circle of reference should be used in the computation as in the experiment. This, however, has not been done by previous writers. In their computations they used as their circle of reference on the level surface of $R$, not the circle of latitude of $R$, but that of the point $T$ in which the line $t$ pierces the level surface of $R$. That a great error has thus been made, appears in Assumption 4, below, in which it is shown that for a second approximation to the potential function the distance (measured along a meridian) between the circles of latitude of $R$ and $T$ (the part which has been neglected) is eighteen times as great as that between $T$ and $S$ (the part which has been computed), $S$ being the point in which the body, which falls from $P_0$, strikes the level surface of $R$. For a first approximation to the potential function, however, the distance between the circles of latitude of $R$ and $T$ may be small in comparison to that between $T$ and $S$.

Now let us compare the assumptions and the results of previous writers, who disregarded either one or both of the facts set forth in I and II, with those of this paper, in which these facts are not disregarded. In order to do this it is desirable first to give a brief description of the method of this paper.

## Statement and Comparison of Results.

After the falling body is released from its initial position, $P_0$, of rest with respect to the rotating earth, it moves under the influence of a field of force $F_1$ which is fixed in space.† A plumb-line $P_0R$ at rest is in equilibrium under the influence of a field of force $F_2$ which is fixed with respect to the rotating earth. If $U_1 = f_1(r, z)$ and $U_2 = f_2(r, z)$ (where $r$ represents the distance of a general point from the earth's axis of rotation $OZ$, and $z$ that from a plane $\pi$ perpendicular to $OZ$ at $O$, and where $O$ is taken at the earth's center, although this is not necessary) are potential functions of the fields $F_1$ and $F_2$ respectively, there exists between them the relation

$$U_1 + \tfrac{1}{2}\omega^2 r^2 = U_2,$$

in which $\omega$ denotes the angular velocity of the earth's rotation.

The path of the falling body, which has received an initial velocity from the rotating earth, is a curve $c$ in the fixed field $F_1$. The curve $c$ lies on a surface of revolution whose axis coincides with that of the earth. Let us denote by $c_1$

* It is assumed that the plumb-bob is a heavy particle and that the line is weightless and perfectly flexible.

† That is, fixed with respect to a set of axes whose origin is the center of the earth and whose directions are fixed directions in space.

that meridian curve of this surface of revolution which lies in the plane of $OZ$ and $P_0$. The locus of the plumb-bobs $R$ of all plumb-lines which are supported at $P_0$, is a curve $c_2$ which also lies in the plane of $OZ$ and $P_0$. It will be shown later that the curves $c_1$ and $c_2$ are tangent at $P_0$. The line $t$ defined above is their common tangent.

Let us now take the common tangent and the common normal to these curves as a pair of rectangular axes and refer the equations of curves $c_1$ and $c_2$ to them. The coördinate which is measured along the tangent from the point $P_0$ we will denote by $\xi$ and regard as positive when measured in the direction of the earth's center. The other coördinate we will denote by $\eta$ and regard as positive when measured in the direction of the north pole. (See Fig. 4.) The equations of $c_1$ and $c_2$ then become

$$\eta_1 = -\frac{1}{6}\frac{C + B_1\omega^2 + A_1\omega^4}{D^{\frac{3}{2}}}\xi^2 + \cdots, \tag{1}$$

$$\eta_2 = -\frac{C + B_2\omega^2 + A_2\omega^4}{D^{\frac{3}{2}}}\xi^2 + \cdots, \tag{2}$$

where

$$C = \left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0\left[\left(\frac{\partial f_1}{\partial r}\right)_0^2 - \left(\frac{\partial f_1}{\partial z}\right)_0^2\right] + \left[\left(\frac{\partial^2 f_1}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0\right]\left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$B = 2\left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0\left(\frac{\partial f_1}{\partial r}\right)_0 r_0 + \left[\left(\frac{\partial^2 f_1}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0\right]\left(\frac{\partial f_1}{\partial z}\right)_0 r_0,$$

$$A = \left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0 r_0^2,$$

$$B_1 = B + 3\left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0, \qquad B_2 = B - \left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$A_1 = A + 3\left(\frac{\partial f_1}{\partial z}\right)_0 r_0, \qquad A_2 = A - \left(\frac{\partial f_1}{\partial z}\right)_0 r_0,$$

$$D = \left(\frac{\partial f_1}{\partial r}\right)_0^2 + \left(\frac{\partial f_1}{\partial z}\right)_0^2 + 2\left(\frac{\partial f_1}{\partial r}\right)_0 r_0\omega^2 + r_0^2\omega^4,$$

the subscript $(_0)$ indicating that the derivatives of $f_1$ have been computed for the point $P_0(r = r_0, z = z_0)$.

The southerly deviation is the difference $\eta_2 - \eta_1$, when we substitute for $\xi$ the height $h = P_0R$ through which the body falls, and for $f_1$ a sufficiently close approximation to the potential function of the earth's gravitational field of force.

The southerly deviation can also be expressed in the following very simple form*

* This formula was presented to the Society (Chicago Section) April 29, 1911.

$$\frac{4\omega^2 \sin\phi\cos\phi + 5\left(\frac{\partial g}{\partial \eta}\right)_0}{6g_0} h^2, \tag{3}$$

in which $g_0$ and $\phi$ denote the acceleration and the astronomical latitude, respectively, at the point $P_0$. For,

$$g_0 = \sqrt{\left(\frac{\partial f_2}{\partial z}\right)_0^2 + \left(\frac{\partial f_2}{\partial r}\right)_0^2} = \sqrt{D}, \qquad g_0 \sin\phi = \left(\frac{\partial f_2}{\partial z}\right)_0 = \left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$g_0 \cos\phi = \left(\frac{\partial f_2}{\partial r}\right)_0 = \left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0.$$

By the equation of transformation in § 7,

$$U_1 = f_1(r, z) = f_1(-\xi\cos\phi - \eta\sin\phi + r_0, -\xi\sin\phi + \eta\cos\phi + z_0),$$

and therefore

$$\frac{\partial^2 U_1}{\partial\eta\,\partial\xi} = (\sin^2\phi - \cos^2\phi)\frac{\partial^2 f_1}{\partial r\,\partial z} + \sin\phi\cos\phi\left[\frac{\partial^2 f_1}{\partial r^2} - \frac{\partial^2 f_1}{\partial z^2}\right].$$

On the other hand,

$$\frac{\partial^2 U_1}{\partial\eta\,\partial\xi} = \frac{\partial^2 U_2}{\partial\eta\,\partial\xi} - \omega^2\sin\phi\cos\phi.$$

Hence equations (1) and (2) assume the following forms:

$$\eta_1 = \frac{1}{6g_0}\left[\left(\frac{\partial^2 U_2}{\partial\eta\,\partial\xi}\right)_0 - 4\omega^2\sin\phi\cos\phi\right]\xi^2, \qquad \eta_2 = \frac{1}{g_0}\left(\frac{\partial^2 U_2}{\partial\eta\,\partial\xi}\right)_0 \xi^2,$$

and

$$\eta_2 - \eta_1 = \frac{1}{6g_0}\left[4\omega^2\sin\phi\cos\phi + 5\left(\frac{\partial^2 U_2}{\partial\eta\,\partial\xi}\right)_0\right]\xi^2.$$

But

$$\left(\frac{\partial^2 U_2}{\partial\eta\,\partial\xi}\right)_0 = \left[\frac{\partial}{\partial\eta}\left(\frac{\partial U_2}{\partial\xi}\right)\right]_0 = \left(\frac{\partial g}{\partial\eta}\right)_0,$$

and hence formula (3).

It will be observed that the coefficient of $\xi^2$ in each of the equations (1) and (2) contains the second as well as the first derivatives of $f_1$. It is for this reason that the potential function must be developed to terms of the second order inclusive in the neighborhood of $P_0(r_0, z_0)$.

In the discussion which follows we shall call $\eta_1$ and $\eta_2$ the parts of the southerly deviation which are contributed by the trajectory and the plumb-bob locus respectively. It is the part $\eta_2$ which has been neglected, and the part $\eta_1$ which has been computed, by previous writers. We shall compare these for each of several forms of $f_1$, we shall also express the results corresponding to different forms of $f_1$ in terms of the same unit.

## 1. *The Assumption of Gauss.*

Gauss assumed that the force to which a body, at rest with respect to the rotating earth, is subjected is constant in magnitude and direction. That is,*

$$f_2(r, z) = -g\cos\phi \cdot r - g\sin\phi \cdot z,$$

and therefore

$$f_1(r, z) = f_2(r, z) - \tfrac{1}{2}\omega^2 r^2,$$

in which $\phi$ and $g$ are the astronomical latitude and the acceleration at $P_0$. For this potential function equations (1) and (2) become †

$$\eta_1 = -\tfrac{2}{3}\sin\phi\cos\phi \cdot \frac{\omega^2}{g}\cdot\xi^2, \qquad \eta_2 = 0\cdot\xi^2,$$

and

$$\eta_2 - \eta_1 = -\eta_1 = \tfrac{2}{3}\sin\phi\cos\phi\cdot\frac{\omega^2}{g}\cdot\xi^2.$$

Thus we see that in this case the *whole southerly deviation is due to the trajectory,* since the plumb-bob locus coincides with the tangent $t$. (See Fig. 1.) The curve $c_1$ lies to the south of the tangent $t$.

The potential function used by Gauss is a first approximation to that of the earth's field. By statement I this is not a sufficiently close approximation.

Before making the next assumption let us express the result just obtained in a form in which it may be compared with the results which follow.

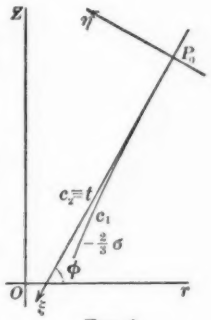

FIG. 1.

With sufficient approximation

$$\sin\phi = \frac{z_0}{\rho_0}, \qquad \cos\phi = \frac{r_0}{\rho_0}, \qquad \frac{\omega^2}{g} = \frac{\sigma}{\rho_0},$$

where ‡

$$\sigma = \frac{\omega^2\rho_0}{M/\rho_0^2}, \qquad \rho_0 = \sqrt{r_0^2 + z_0^2},$$

and $M$ represents the mass of the earth. Then

* Strictly speaking this function does not quite represent Gauss' assumption. Under Gauss' assumption the equipotential surfaces of the field $F_2$ are parallel planes, while the equipotential surfaces corresponding to the function $f_2$ here assumed are parallel cones of revolution which have as a common axis that of the earth.

† The formula given by GAUSS is

$$\tfrac{1}{6}\sin\phi\cos\phi\, g\omega^2 t^4.$$

But since

$$\xi = \tfrac{1}{2}gt^2 + \cdots,$$

we see that the result is the same as the one just found. See GAUSS' *Werke,* vol. 5 (1867), p. 502. GAUSS' formula can also be obtained by putting $(\partial g/\partial\eta)_0 = 0$ in formula (3).

‡ The value of $\sigma$ is about $\frac{1}{289}$.

$$\eta_2 - \eta_1 = -\eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \cdot \tfrac{2}{3}\sigma \cdot \xi^2.$$

## 2. *A Second Assumption.*

Let us now assume that the field of force $F_1$ is central, the center being at the earth's center and the law of attraction that of Newton. Then *

$$f_1(r, z) = \frac{M}{\rho}, \qquad \rho = \sqrt{r^2 + z^2},$$

where $M$ represents the mass of the earth. For this potential function equations (1) and (2) become †

$$\eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0}\left[0 + 0 + \frac{1}{2}\frac{z_0^2}{\rho_0^2}\sigma^2 + \cdots\right]\xi^2,$$

$$\eta_2 = \frac{r_0 z_0}{\rho_0} \cdot \frac{1}{\rho_0}\left[0 + 4\sigma + \frac{8r_0^2 - z_0^2}{\rho_0^2}\sigma^2 + \cdots\right]\xi^2,$$

and

$$\eta_2 - \eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0}\left[0 + 4\sigma + \frac{16r_0^2 - 3z_0^2}{2\rho_0^2}\sigma^2 + \cdots\right]\xi^2 + \cdots.$$

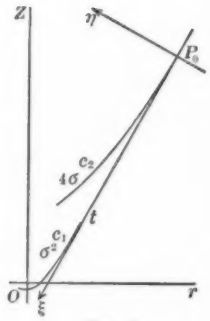


FIG. 2.

In this case *practically the whole southerly deviation is due to the plumb-bob locus*, since in the equation of the curve $c_1$ the coefficient of $\xi^2$ is an infinitesimal of second order with respect to the coefficient of $\xi^2$ in the equation of $c_2$. (See Fig. 2.) Both of the curves $c_1$ and $c_2$ lie to the north of the tangent $t$.

Since the curve $c_1$ lies to the north of $t$, it follows that those computers who neglect the part $\eta_2$ are forced to say that, for this potential function, the falling body has a *northerly*, and not a southerly, deviation.

## 3. *A Third Assumption.*

Let us assume, as above, that the field of force $F_1$ is central, the center being at the earth's center. Here we will assume, not the law of Newton, but that the

* For this potential function the level (equipotential) surface of the field $F_2$ which passes through the plumb-bob $R$, is an ellipsoid of oblateness $\frac{1}{2}\sigma$. [Put $\epsilon = 0$ in the assumption 4.]

† In this case the curve $c_1$ is a conic of which $O\,(r = 0,\ z = 0)$ is a focus. This follows from the fact that the curve $c$ is an ellipse with focus at $O$.

magnitude of the force varies directly as the distance from the center. Then *

$$f_1(r, z) = -\frac{K}{2}\rho^2, \qquad \rho^2 = r^2 + z^2,$$

where $K$ is a positive constant. For this potental function equations (1) and (2) become †

$$\eta_1 = -\tfrac{1}{2}\frac{r_0 z_0}{\rho_0^2}\cdot\frac{1}{\rho_0}\left[0 + \frac{\omega^2}{K} + \frac{2r_0^2 - z_0^2}{\rho_0^2}\frac{\omega^4}{K^2}\cdots\right]\xi^2 + \cdots,$$

$$\eta_2 = -2\eta_1 \qquad \text{and} \qquad \eta_2 - \eta_1 = -3\eta_1.$$

In this case $c_1$ falls to the south, and $c_2$ to the north, of $t$. Hence the trajectory contributes $\frac{1}{3}$, and the plumb-bob locus $\frac{2}{3}$, of the southerly deviation. (See Fig. 3.)

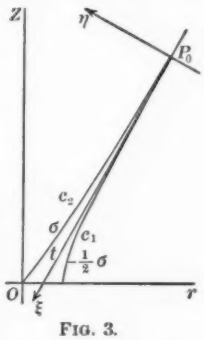

FIG. 3.

If, at the initial point, the forces due to assumptions 2 and 3 be the same, $K\rho_0 = M/\rho_0^2$ and therefore $\omega^2/K = \sigma$.

*The General Differential Equation of Gauss.*

Before making the supposition stated in Assumption 1 above, Gauss found the differential equations of motion (relative to a set of axes fixed with respect to the rotating earth) of a falling body, on the supposition that the field of force $F_1$ is central,‡ the center being the point in which the earth's axis is pierced by the vector which represents the force of the field $F_1$ at the initial point of the falling body. Gauss does not solve these differential equations, in which he denotes by $p$ the magnitude of the force of the field $F_1$ and by $M$ a constant due to air resistance. If we put this $M$ equal to zero (i. e., neglect air resistance) and assume that $p$ varies

* For this potential function all of the level (equipotential) surfaces of the field $F_2$ are ellipsoids of oblateness $\frac{1}{2}\sigma$. Their equations are

$$K(r^2 + z^2) - \omega^2 r^2 = C.$$

Their oblateness is

$$1 - \frac{\rho_p}{\rho_e} = 1 - \frac{\frac{\sqrt{C}}{\sqrt{K}}}{\frac{\sqrt{C}}{\sqrt{K - \omega^2}}} = 1 - \sqrt{1 - \frac{\omega^2}{K}} = \tfrac{1}{2}\frac{\omega^2}{K} = \tfrac{1}{2}\sigma.$$

† In this case the curve $c_1$ is an hyperbola whose center is at $O$ and whose foci are on the $r$-axis; the curve $c_2$ is an equilateral hyperbola which passes through $O$ and whose asymptotes are parallel to the axes of $r$ and $z$. The truth of the first part of this statement follows from the fact that the curve $c$ is an ellipse whose center is at $O$. The curve $c_2$ is of the form mentioned since it is the locus of the feet of perpendiculars dropped from a point to a family of similar and concentric ellipses.

‡ At least if $q$ is constant in his equations [3].

inversely as the square of the distance (or directly as the distance) from the center ($X=0$, $Y=0$, $Z=q$) of Gauss' field $F_1$, we obtain a set of differential equations, the solution of which should yield for the $x$ of Gauss the same value as that obtained by replacing $z_0$ by $z_0+\alpha$ in the expression given for $\eta_1$ under Assumption 2 (or 3), $\alpha$ being the distance of the center of Gauss' field $F_1$ to the south from the earth's center $O$.

### 4. *A Fourth Assumption.*

Now let us assume that the potential function $f_1$ has the form *:

$$f_1(r,z)=\frac{M}{\rho}+\frac{\epsilon}{3}M\rho_1^2\frac{r^2-2z^2}{\rho^5},$$

in which $\rho$ and $M$ have the same meaning as in assumption 2. The symbol $\rho_1$ represents the mean radius of the Standard Spheroid, and

$$\epsilon=e_1-\tfrac{1}{2}\sigma_1,$$

where $e_1$ represents the ellipticity (or oblateness) of the Standard Spheroid, and

$$\sigma_1=\frac{\omega^2\rho_1}{M/\rho_1^2}.$$

The values of the constants are †

$$\sigma_1=\frac{1}{288.38}=.003468 \qquad \text{and} \qquad e_1=\frac{1}{293.5}=.003407,$$

$$\epsilon=.00167.$$

For this form of $f_1$, equations (1) and (2) become

$$\eta_1=-\frac{1}{6}\frac{r_0z_0}{\rho_0^2}\cdot\frac{1}{\rho_0}\left[2\left(\frac{\rho_1}{\rho_0}\right)^2\epsilon+\left(\frac{\rho_1}{\rho_0}\right)^4\frac{16z_0^2-10r_0^2}{\rho_0^2}\epsilon^2-2\left(\frac{\rho_1}{\rho_0}\right)^2\frac{r_0^2}{\rho_0^2}\epsilon\sigma-3\frac{z_0^2}{\rho_0^2}\sigma^2\cdots\right]\xi^2,$$

$$\eta_2=-\frac{r_0z_0}{\rho_0^2}\cdot\frac{1}{\rho_0}\left[2\left(\frac{\rho_1}{\rho_0}\right)^2\epsilon-4\sigma+\left(\frac{\rho_1}{\rho_0}\right)^4\frac{16z_0^2-10r_0^2}{\rho_0^2}\epsilon^2-6\left(\frac{\rho_1}{\rho_0}\right)^2\frac{r_0^2}{\rho_0^2}\epsilon\sigma+\frac{z_0^2-8r_0^2}{\rho_0^2}\sigma^2\cdots\right]\xi^2,$$

and

$$\eta_2-\eta_1=\frac{r_0z_0}{\rho_0^2}\cdot\frac{1}{\rho_0}\left[-\frac{5}{6}\left(\frac{\rho_1}{\rho_0}\right)^2 2\epsilon+4\sigma-\frac{5}{6}\left(\frac{\rho_1}{\rho_0}\right)^4\frac{16z_0^2-10r_0^2}{\rho_0^2}\epsilon^2+\frac{17}{3}\left(\frac{\rho_1}{\rho_0}\right)^2\frac{r_0^2}{\rho_0^2}\epsilon\sigma+\frac{16r_0^2-3z_0^2}{2\rho_0^2}\sigma^2\cdots\right]\xi^2.$$

The ratio $\rho_1/\rho_0$ is nearly equal to unity and $2\epsilon(=.00334)$ is nearly equal to $\sigma(=.00345)$. Hence, approximately, the above equations become

* This function may be regarded as a sufficiently close approximation to the potential function for which the Standard Spheroid is an equipotential surface.

† For the derivation of this function and the values of $e_1$ and $\sigma_1$ see POINCARÉ, *Figures d'Equilibre d'une Masse Fluide* (1902), Chap. V.

$$\eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[ -\tfrac{1}{6}\sigma \right] \xi^2, \qquad \eta_2 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[ 3\sigma \right] \xi^2,$$

and

$$\eta_2 - \eta_1 = \frac{r_0 z_0}{\rho_0^2} \frac{1}{\rho_0} \left[ \tfrac{19}{6}\sigma \right] \xi^2.$$

In this case $c_1$ falls to the south, and $c_2$ to the north. of $t$. The trajectory contributes $\frac{1}{19}$, and the plumb-bob locus $\frac{18}{19}$, to the southerly deviation. (See Fig. 4.)

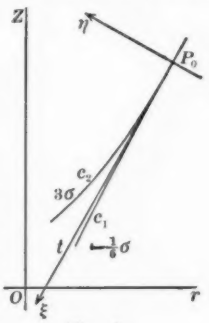

FIG. 4.

It will be observed that the potential function and the results under the second assumption may be obtained from those under this assumption by setting $\epsilon$ equal to zero.

The preceding results and figures 1, 2, 3 and 4 clearly show that the curves $c_1$ (or $c_2$) which correspond to the different assumptions are not even approximations to one another. The four potential functions assumed above may be regarded as having equal first derivatives* at $P_0(r_0, z_0)$. But in order that the curves $c_1$ (or $c_2$) which correspond to two potential functions $f_1$, shall osculate at $P_0$, both the first and second derivatives of these two potential functions must be equal at $P_0$. This follows from the fact that the coefficient of $\xi^2$ in each of equations (1) and (2) involves both the first and second derivatives of $f_1$.

### *The Work of M. le Comte de Sparre.*†

M. le Comte de Sparre computes the portion of the southerly deviation which is denoted by $\eta_1$ in this paper. In his first paper he assumes potential functions which are practically identical with those of Assumptions 2 and 3. His results, under these assumptions, are

$$x = -\frac{1}{2} \frac{R\omega^4 h^2}{g^2} \sin^3 \lambda \cos \lambda \quad \text{and} \quad x = \frac{1}{2} \frac{\omega^2 h^2}{g} \sin \lambda \cos \lambda,$$

respectively, and those of this paper are

$$\eta_1 = \frac{1}{2} \frac{r_0 z_0}{\rho_0^2} \cdot \frac{z_0^2}{\rho_0^2} \cdot \frac{1}{\rho_0} \sigma^2 \xi^2 \quad \text{and} \quad \eta_1 = -\frac{1}{2} \frac{r_0 z_0}{\rho_0^2} \frac{1}{\rho_0} \left[ \sigma + (\ )\sigma^2 \right] \xi^2,$$

respectively. Since $x$ is positive for southerly, and $\eta_1$ is positive for northerly deviations, and since

* That is, the vector which represents the force of the field $F_1$ at the initial point of the falling body in $F_1$, may be taken to be the same for all the assumptions. As a consequence of this the line $t$ will be the same for all the assumptions.

† Bulletin de la Société Mathématique de France, vol. 33 (1905). First paper, page 65, second paper, page 146.

$$R = \rho_0, \qquad \sin\lambda = \frac{z_0}{\rho_0}, \qquad \cos\lambda = \frac{r_0}{\rho_0}, \qquad h^2 = \xi^2, \qquad R\frac{\omega^2}{g} = \sigma,$$

it follows that the two sets of formulas are identical.

In the second paper a potential function is chosen which is practically equivalent to that of Assumption 4. The result of M. le Compte de Sparre is

$$x = \frac{\omega^2}{23}\sin\lambda\cos\lambda\, gt^4$$

and that of this paper is

$$\eta_1 = \frac{r_0 z_0}{\rho_0^2}\frac{1}{\rho_0}(-\tfrac{1}{6}\sigma\cdots)\xi^2.$$

Since $h = \xi = \frac{1}{2}gt^2 + \cdots$, and by the above substitution, these formulas may be written in the following forms:

$$x = \tfrac{4}{23}\sin\lambda\cos\lambda\frac{\sigma}{R}h^2 \qquad \text{and} \qquad -\eta_1 = +\tfrac{1}{6}\sin\lambda\cos\lambda\frac{\sigma}{R}h^2.$$

Therefore these expressions are nearly the same.

*Conclusion.*

Of the facts set forth in statements I and II of the introduction, the assumption of Gauss disregarded the first but not the second, and the assumption of the second paper of M. le Comte de Sparre disregarded the second but not the first, while the two assumptions of the first paper of M. le Comte de Sparre disregarded both the first and the second. On the other hand the diference $\eta_2 - \eta_1$ for assumption 4 was obtained with due regard for the facts set forth in Statements I and II. This difference yields results for the southerly deviation which are several times as great as those obtained from the formulas of Gauss and M. le Compte de Sparre. It is not necessary to retain all the terms which are included in the expression for this difference, since $\sigma^2$, $\epsilon^2$ and $\epsilon\sigma$ are small in comparison with $\sigma$ and $\epsilon$. Since the ratio $\rho_1/\rho_0$ is nearly equal to unity, and therefore

$$\sigma\left(=\frac{\omega^2\rho_0}{M/\rho_0^2}\right) \qquad \text{and} \qquad \sigma_1\left(=\frac{\omega^2\rho_1}{M/\rho_1^2}\right)$$

are nearly equal, the expression for the southerly deviation of a falling body assumes the form:

$$\tfrac{1}{2}\sin 2\phi_0(4\sigma_1 - \tfrac{5}{3}\epsilon)\frac{h^2}{\rho_0},$$

where $\phi_0$ denotes the geocentric latitude of the initial point $P_0$, $\rho_0$ the distance from the earth's center to $P_0$, $h$ the height through which the body falls, and $\sigma_1$ and $\epsilon$ the numerical constants given in Assumption 4.

### *A Numerical Example.*

For the data of Benzenberg's experiment in St. Michael's Tower, namely $\phi = 53° 33'$, $h = 235$ feet, the formula of Gauss ( $\frac{2}{3} \sin \phi \cos \phi \omega^2 h^2/g$ ) yields the value .00046 lines*, and the formula of this paper [ $\frac{1}{2} \sin 2\phi_0 (4\sigma_1 - \frac{5}{3}\epsilon) h^2/\rho_0$, where $\phi_0$ denotes the geocentric latitude of $P_0$ and equals $53° 22'$ if the astronomical latitude is $\phi = 53° 33'$ ] yields the value .00202 lines. The latter result is about four and one-half times as great as the former, as we knew it should be from a comparison of the expressions for $\eta_2 - \eta_1$ under assumptions 1 and 4.

### *The Results of Experiments.*

In experiments for the determination of the southerly deviation of falling bodies, results are obtained which are many times greater than those accounted for by any of the formulas given above.†

Since the greater part of the southerly deviation is due to the plumb-bob locus, it occurs to the author that experiments for the determination of the deviation between a very long plumb-line and a short plumb-line, which are supported at the same point, might give results of greater value than those for the determination of the southerly deviation of a falling body. The annoying factor of air resistance occurs in the latter, but not in the former. By enclosing the plumb-lines in an air-tight tube and making the observations through glass windows in the tube, air currents could also be eliminated.

## The Establishment of General Equations (1) and (2).

§ 1. *The curve c.* The curve $c$ has already been defined as the path of the falling body in the fixed field $F_1$. In order to get its equations let us choose a set of rectangular axes in $F_1$ and denote by $x_1$, $y_1$, $z_1$ the coördinates of a general point referred to these axes, the origin of which is taken at the earth's center $O$. The axis of $z_1$ is taken coincident with the earth's axis of rotation and is regarded as positive in the direction of the earth's north pole. The axes of $x_1$ and $y_1$ then lie in the earth's equatorial plane. The direction of the axis of $x_1$ is so chosen that the plane $x_1 z_1$ contains the position $\bar{P}_0$ which the falling body occupied at the instant when its connection with the rotating earth was severed, and its sense is such that the coördinate of this initial point is positive. The axis of $y_1$, which is perpendicular to that of $x_1$, is positive in the direction in which the earth rotates at the initial point. (See Fig. 5.)

* 144 lines = 1 foot.

† See the following papers: Professor Florian Cajori; *The Unexplained Southerly Deviation of Falling Bodies*, Science, Vol. 14 (1901), p. 853. Professor Edwin H. Hall, *Do Falling Bodies Move South?* Physical Review, Vol. 17 (1903), Nos. 3, 4.

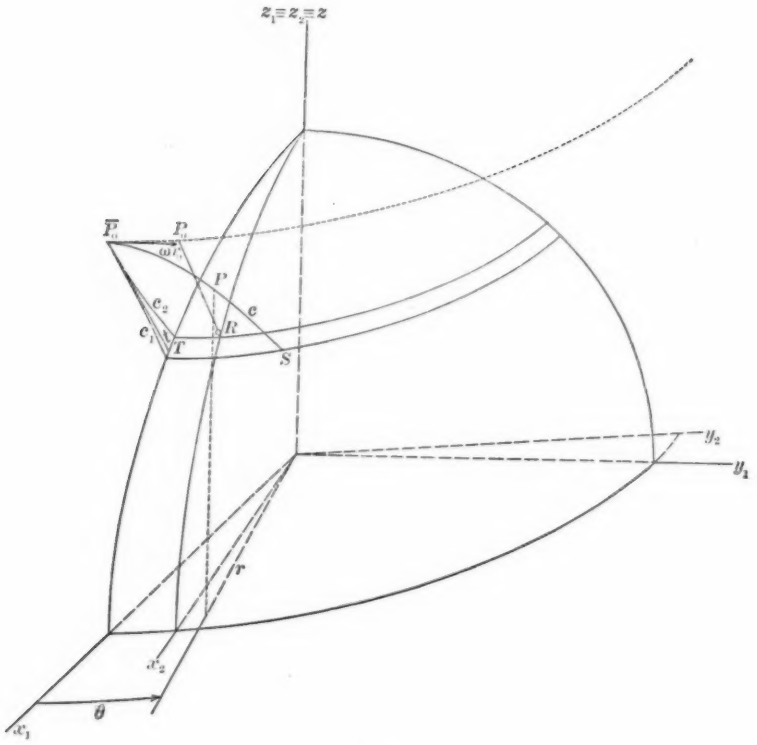


FIG. 5.

If, as above,

$$U_1(x_1, y_1, z_1) = f_1(r, z)$$

represents the potential function of the field $F_1$, and $\omega$ the angular velocity of the earth's rotation, then the curve $c$ is that integral curve of the differential equations

$$\frac{d^2x_1}{dt^2} = \frac{\partial U_1}{\partial x_1}, \qquad \frac{d^2y_1}{dt^2} = \frac{\partial U_1}{\partial y_1}, \qquad \frac{d^2z_1}{dt^2} = \frac{\partial U_1}{\partial z_1},$$

which, at $t = 0$, is subject to the conditions

$$x_1 = r_0, \qquad y_1 = 0, \qquad z_1 = z_0,$$

$$\frac{dx_1}{dt} = 0, \qquad \frac{dy_1}{dt} = \omega r_0, \qquad \frac{dz_1}{dt} = 0.$$

§ 2. *The curve* $c_1$. The curve $c_1$ has already been defined as the meridian curve of the surface of revolution, of axis $z_1$, on which the curve $c$ lies. It may also be defined as the path of the falling body in the moving plane which is determined by the body and the axis of $z_1$. In order to get its equation let us make the following transformation of coördinates:

$$x_1 = r\cos\theta, \qquad y_1 = r\sin\theta, \qquad z_1 = z,$$

in which $\theta$ denotes the angle which the plane determined by the falling body and the axis of $z_1$ makes with the plane $x_1 z_1$. (See Fig. 5.) The preceding differential equations are then transformed into the following differential equations:

$$r'' - r\theta'^2 = \frac{\partial f_1}{\partial r},$$

$$r\theta'' + 2r'\theta' = 0,$$

$$z'' = \frac{\partial f_1}{\partial z},$$

where the primes (′) and seconds (″) denote the first and second derivatives, respectively, with regard to the time $t$. The second of these equations on integration becomes

$$\theta' = \frac{k}{r^2},$$

where $k$ is a constant of integration, and hence reduces the first to

$$r'' = \frac{k^2}{r^3} + \frac{\partial f_1}{\partial r}.$$

The initial conditions assume the following form: when $t = 0$,

$$r = r_0, \qquad \theta = 0, \qquad z = z_0,$$
$$r' = 0, \qquad \theta' = \omega, \qquad z' = 0.$$

Hence the constant $k$ is given by the relation

$$\omega = k/r_0^2.$$

Therefore the curve $c_1$ is that integral curve of the differential equations:

$$r'' = \frac{\omega^2 r_0^4}{r^3} + \frac{\partial f_1}{\partial r}, \qquad z'' = \frac{\partial f_1}{\partial z},$$

which, at $t = 0$, is subject to the conditions

$$r = r_0, \qquad z = z_0, \qquad r' = 0, \qquad z' = 0.$$

If we put

$$V(r, z) = -\frac{\omega^2 r_0^4}{2r^2} + f_1(r, z),$$

the last differential equations may be written in the following form:

$$r'' = \frac{\partial V}{\partial r}, \qquad z'' = \frac{\partial V}{\partial z}.$$

§ 3. *The curve* $c_2$. The curve $c_2$ has already been defined as the locus of plumb-bobs of all plumb-lines which are supported at the initial point, $P_0$, of the falling body in the field $F_2$. It is therefore the locus of the points of tangency of tangent lines which are drawn from $P_0$ to the lines of force of the field $F_2$. In order to get its equation let us choose a set of rectangular axes in $F_2$ and denote by $x_2, y_2, z_2$ the coördinates of a general point referred to these axes, the origin of which is also taken at the earth's center $O$. The axis of $z_2$ is taken, just as that of $z_1$, coincident with the earth's axis of rotation and is regarded as positive in the direction of the earth's north pole. The axes of $x_2$ and $y_2$ lie in the earth's equatorial plane, and when $t = 0$, they coincide with the axes of $x_1$ and $y_1$, respectively. (See Fig. 5.) The point $P_0$ lies in the plane $x_2 z_2$. If, as above,

$$U_2(x_2, y_2, z_2) = f_2(r, z),$$

where

$$r = \sqrt{x_2^2 + y_2^2} \qquad \text{and} \qquad z = z_2,$$

represents the potential function of the field $F_2$, the equations of $c_2$ are

$$\frac{x_2 - r_0}{\frac{\partial U_2}{\partial x_2}} = \frac{y_2 - 0}{\frac{\partial U_2}{\partial y_2}} = \frac{z_2 - z_0}{\frac{\partial U_2}{\partial z_2}}.$$

But since

$$\frac{\partial U_2}{\partial x_2} = \frac{\partial f_2}{\partial r} \cdot \frac{x_2}{r}, \qquad \frac{\partial U_2}{\partial y_2} = \frac{\partial f_2}{\partial r} \cdot \frac{y_2}{r}, \qquad \frac{\partial U_2}{\partial z_2} = \frac{\partial f_2}{\partial z},$$

the curve lies in the plane $y_2 = 0$, in which it has the equation

$$\frac{r - r_0}{\frac{\partial f_2}{\partial r}} = \frac{z - z_0}{\frac{\partial f_2}{\partial z}}.$$

§ 4. *The equation of curve* $c_1$ *in powers of* $r - r_0$.

That solution of the differential equations

$$r'' = \partial V / \partial r, \qquad z'' = \partial V / \partial z$$

which, at $t = 0$, is subject to the conditions $r = r_0, z = z_0, r' = 0, z' = 0$, is

of the form

$$r - r_0 = \alpha_2 t^2 + \alpha_4 t^4 + \cdots, \qquad z - z_0 = \beta_2 t^2 + \beta_4 t^4 + \cdots,$$

where

$$\alpha_2 = \frac{1}{2}\left(\frac{\partial V}{\partial r}\right)_0, \qquad \beta_2 = \frac{1}{2}\left(\frac{\partial V}{\partial z}\right)_0,$$

$$\alpha_4 = \frac{1}{4!}\left\{\left(\frac{\partial^2 V}{\partial r^2}\right)_0\left(\frac{\partial V}{\partial r}\right)_0 + \left(\frac{\partial^2 V}{\partial r \partial z}\right)_0\left(\frac{\partial V}{\partial z}\right)_0\right\},$$

$$\beta_4 = \frac{1}{4!}\left\{\left(\frac{\partial^2 V}{\partial r \partial z}\right)_0\left(\frac{\partial V}{\partial r}\right)_0 + \left(\frac{\partial^2 V}{\partial z^2}\right)_0\left(\frac{\partial V}{\partial z}\right)_0\right\},$$

the subscript $(_0)$ indicating that the derivatives of $V$ have been computed for the point $P_0(r_0, z_0)$. By the elimination of $t$ from these equations, we obtain the equation

$$z - z_0 = a_1(r - r_0) + a_2(r - r_0)^2 + \cdots,$$

where

$$a_1 = \frac{\left(\frac{\partial V}{\partial z}\right)_0}{\left(\frac{\partial V}{\partial r}\right)_0},$$

$$a_2 = \frac{1}{6}\frac{\left(\frac{\partial^2 V}{\partial r \partial z}\right)_0\left\{\left(\frac{\partial V}{\partial r}\right)_0^2 - \left(\frac{\partial V}{\partial z}\right)_0^2\right\} + \left\{\left(\frac{\partial^2 V}{\partial z^2}\right)_0 - \left(\frac{\partial^2 V}{\partial r^2}\right)_0\right\}\left(\frac{\partial V}{\partial r}\right)_0\left(\frac{\partial V}{\partial z}\right)_0}{\left(\frac{\partial V}{\partial r}\right)_0^3}.$$

### § 5. *The equation of curve $c_2$ in powers of $r - r_0$.*

The equation of curve $c_2$, whose finite equation has already been found in § 3, may also be written in the following form:

$$z - z_0 = b_1(r - r_0) + b_2(r - r_0)^2 + \cdots,$$

where

$$b_1 = \left(\frac{z}{dr}\right)_0, \qquad b_2 = \frac{1}{2}\left(\frac{d^2 z}{dr^2}\right)_0.$$

In order to do this let us first write its equation in the form

$$F(r, z) = 0,$$

where

$$F(r, z) = \frac{\partial f_2}{\partial z}(r - r_0) - \frac{\partial f_2}{\partial r}(z - z_0).$$

The derivatives $dz/dr$ and $d^2z/dr^2$ are expressible in terms of the derivatives of $F$ by means of the relations:

$$\frac{\partial F}{\partial r}+\frac{\partial F}{\partial z}\cdot\frac{dz}{dr}=0,$$

$$\frac{\partial^2 F}{\partial r^2}+2\frac{\partial^2 F}{\partial r\partial z}\cdot\frac{dz}{dr}+\frac{\partial^2 F}{\partial z^2}\left(\frac{dz}{dr}\right)^2+\frac{\partial F}{\partial z}\cdot\frac{d^2z}{dr^2}=0,$$

and the derivatives of $F$, in terms of those of $f_2$, by means of the relations:

$$\frac{\partial F}{\partial r}=\frac{\partial^2 f_2}{\partial r\partial z}(r-r_0)-\frac{\partial^2 f_2}{\partial r^2}(z-z_0)+\frac{\partial f_2}{\partial z},$$

$$\frac{\partial F}{\partial z}=\frac{\partial^2 f_2}{\partial z^2}(r-r_0)-\frac{\partial^2 f_2}{\partial r\partial z}(z-z_0)-\frac{\partial f_2}{\partial r},$$

$$\frac{\partial^2 F}{\partial r^2}=\frac{\partial^3 f_2}{\partial r^2\partial z}(r-r_0)-\frac{\partial^3 f_2}{\partial r^3}(z-z_0)+2\frac{\partial^2 f_2}{\partial r\partial z},$$

$$\frac{\partial^2 F}{\partial r\partial z}=\frac{\partial^3 f_2}{\partial r\partial z^2}(r-r_0)-\frac{\partial^3 f_2}{\partial r^2\partial z}(z-z_0)+\frac{\partial^2 f_2}{\partial z^2}-\frac{\partial^2 f_2}{\partial r^2},$$

$$\frac{\partial^2 F}{\partial z^2}=\frac{\partial^3 f_2}{\partial z^3}(r-r_0)-\frac{\partial^3 f_2}{\partial r\partial z^2}(z-z_0)-2\frac{\partial^2 f_2}{\partial r\partial z},$$

and therefore

$$\left(\frac{\partial F}{\partial r}\right)_0=\left(\frac{\partial f_2}{\partial z}\right)_0,$$

$$\left(\frac{\partial F}{\partial z}\right)_0=-\left(\frac{\partial f_2}{\partial r}\right)_0,$$

$$\left(\frac{\partial^2 F}{\partial r^2}\right)_0=2\left(\frac{\partial^2 f_2}{\partial r\partial z}\right)_0,$$

$$\left(\frac{\partial^2 F}{\partial r\partial z}\right)_0=\left(\frac{\partial^2 f_2}{\partial z^2}\right)_0-\left(\frac{\partial^2 f_2}{\partial r^2}\right)_0,$$

$$\left(\frac{\partial^2 F}{\partial z^2}\right)_0=-2\left(\frac{\partial^2 f_2}{\partial r\partial z}\right).$$

Hence we find

$$b_1=\left(\frac{dz}{dr}\right)_0=-\frac{\left(\frac{\partial F}{\partial r}\right)_0}{\left(\frac{\partial F}{\partial z}\right)_0}=\frac{\left(\frac{\partial f_2}{\partial z}\right)_0}{\left(\frac{\partial f_2}{\partial r}\right)_0},$$

$$b_2=\tfrac{1}{2}\left(\frac{d^2z}{dr^2}\right)_0=\tfrac{1}{2}\frac{\left(\frac{\partial^2 F}{\partial r^2}\right)_0+2\left(\frac{\partial^2 F}{\partial r\partial z}\right)_0\left(\frac{dz}{dr}\right)_0+\left(\frac{\partial^2 F}{\partial z^2}\right)_0\left(\frac{dz}{dr}\right)_0^2}{-\left(\frac{\partial F}{\partial z}\right)_0}$$

$$= \frac{\left(\frac{\partial^2 f_2}{\partial r \partial z}\right)_0 \left\{\left(\frac{\partial f_2}{\partial r}\right)_0^2 - \left(\frac{\partial f_2}{\partial z}\right)_0^2\right\} + \left\{\left(\frac{\partial^2 f_2}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f_2}{\partial r^2}\right)_0\right\}\left(\frac{\partial f_2}{\partial r}\right)_0\left(\frac{\partial f_2}{\partial z}\right)_0}{\left(\frac{\partial f_2}{\partial r}\right)_0^3}.$$

§ 6. *The relation between the curves* $c_1$ *and* $c_2$. In order to find a relation between the curves $c_1$ and $c_2$ let us express the coefficients $a_1$, $a_2$, $b_1$, $b_2$ in terms of the derivatives of $f_1$. We already have the relations:

$$V = f_1 - \frac{\omega^2 r_0^4}{2r^2} \qquad \text{and} \qquad f_2 = f_1 + \frac{\omega^2}{2} r^2,$$

from which the following relations are found:

$$\frac{\partial V}{\partial r} = \frac{\partial f_1}{\partial r} + \frac{\omega^2 r_0^4}{r^3}, \qquad \frac{\partial f_2}{\partial r} = \frac{\partial f_1}{\partial r} + \omega^2 r,$$

$$\frac{\partial V}{\partial z} = \frac{\partial f_1}{\partial z}, \qquad \frac{\partial f_2}{\partial z} = \frac{\partial f_1}{\partial z},$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{\partial^2 f_1}{\partial r^2} - \frac{3\omega^2 r_0^4}{r^4}, \qquad \frac{\partial^2 f_2}{\partial r^2} = \frac{\partial^2 f_1}{\partial r^2} + \omega^2,$$

$$\frac{\partial^2 V}{\partial r \partial z} = \frac{\partial^2 f_1}{\partial r \partial z}, \qquad \frac{\partial^2 f_2}{\partial r \partial z} = \frac{\partial^2 f_1}{\partial r \partial z},$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 f_1}{\partial z^2}, \qquad \frac{\partial^2 f_2}{\partial z^2} = \frac{\partial^2 f_1}{\partial z^2}.$$

Therefore:

$$\left(\frac{\partial V}{\partial r}\right)_0 = \left(\frac{\partial f_2}{\partial r}\right)_0 = \left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0, \qquad \left(\frac{\partial V}{\partial z}\right)_0 = \left(\frac{\partial f_2}{\partial z}\right)_0 = \left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$\left(\frac{\partial^2 V}{\partial r^2}\right)_0 = \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0 - 3\omega^2, \qquad \left(\frac{\partial^2 f_2}{\partial r^2}\right)_0 = \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0 + \omega^2,$$

$$\left(\frac{\partial^2 V}{\partial r \partial z}\right)_0 = \left(\frac{\partial^2 f_2}{\partial r \partial z}\right)_0 = \left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0, \qquad \left(\frac{\partial^2 V}{\partial z^2}\right)_0 = \left(\frac{\partial^2 f_2}{\partial z^2}\right)_0 = \left(\frac{\partial^2 f_1}{\partial z^2}\right)_0.$$

Hence the relations

$$a_1 = b_1 = \frac{\left(\frac{\partial f_1}{\partial z}\right)_0}{\left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0},$$

$$a_2 = \frac{1}{6}\frac{C + B_1\omega^2 + A_1\omega^4}{\left\{\left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0\right\}^3}, \qquad b_2 = \frac{C + B_2\omega^2 + A_2\omega^4}{\left\{\left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0\right\}^3},$$

where

$$C=\left(\frac{\partial^2 f_1}{\partial r\partial z}\right)_0\left\{\left(\frac{\partial f_1}{\partial r}\right)_0^2-\left(\frac{\partial f_1}{\partial z}\right)_0^2\right\}+\left\{\left(\frac{\partial^2 f_1}{\partial z^2}\right)_0-\left(\frac{\partial^2 f_1}{\partial r^2}\right)_0\right\}\left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$B=2\left(\frac{\partial^2 f_1}{\partial r\partial z}\right)_0\left(\frac{\partial f_1}{\partial r}\right)_0 r_0+\left\{\left(\frac{\partial^2 f_1}{\partial z^2}\right)_0-\left(\frac{\partial^2 f_1}{\partial r^2}\right)_0\right\}\left(\frac{\partial f_1}{\partial z}\right)_0 r_0,$$

$$B_1=B+3\left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0, \qquad B_2=B-\left(\frac{\partial f_1}{\partial r}\right)_0\left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$A=\left(\frac{\partial^2 f_1}{\partial r\partial z}\right)_0 r_0^2, \qquad A_1=A+3\left(\frac{\partial f_1}{\partial z}\right)_0 r_0, \qquad A_2=A-\left(\frac{\partial f_1}{\partial z}\right)_0 r_0.$$

Since $a_1=b_1$, it follows that the curves $c_1$ and $c_2$, which lie in the meridian plane of $P_0(r_0, z_0)$ (i. e., in the plane $y_2=0$), have a common tangent * and a common normal at $P_0$.

## § 7. *The General Equations* (1) *and* (2). *The Southerly Deviation.*

Let us now take the common tangent and the common normal to the curves $c_1$ and $c_2$ as a pair of rectangular axes, and obtain the equations of $c_1$ and $c_2$ referred to them. The coördinate which is measured along the tangent from $P_0$ we shall denote by $\xi$, and regard as positive when measured in the direction of the earth's center. The other coördinate we shall denote by $\eta$ and regard as positive when measured in the direction of the north pole. The equations of transformation from the axes of $r$ and $z$ to those of $\xi$ and $\eta$ are

$$r-r_0=-\cos\phi\cdot\xi-\sin\phi\cdot\eta, \qquad z-z_0=-\sin\phi\cdot\xi+\cos\phi\cdot\eta,$$

where

$$\sin\phi=\frac{b_1}{\sqrt{1+b_1^2}}=\frac{a_1}{\sqrt{1+a_1^2}}, \qquad \cos\phi=\frac{1}{\sqrt{1+b_1^2}}=\frac{1}{\sqrt{1+a_1^2}}.$$

By this transformation the equation

$$z-z_0=a_1(r-r_0)+a_2(r-r_0)^2+\cdots$$

* This common tangent is also tangent at $P_0$ to the line of force of the field $F_2$ which passes through $P_0$. For the differential equation of the lines of force of $F_2$ which line in the plane $y_2=0$ is

$$\frac{dz}{dr}=\frac{\dfrac{\partial f_2}{\partial z}}{\dfrac{\partial f_2}{\partial r}},$$

and therefore the slope at $P_0$ of the line of force which passes through $P_0$ is

$$\frac{\left(\dfrac{\partial f_2}{\partial z}\right)_0}{\left(\dfrac{\partial f_2}{\partial r}\right)_0}=\frac{\left(\dfrac{\partial f_1}{\partial z}\right)_0}{\left(\dfrac{\partial f_1}{\partial r}\right)_0+\omega^2 r_0}=a_1=b_1.$$

of the curve $c_1$ assumes the form

$$+\frac{a_2}{1+a_1^2}\xi^2+\frac{2a_1a_2}{1+a_1^2}\xi\eta+\frac{a_1^2a_2}{1+a_1^2}\eta^2-\sqrt{1+a_1^2}\,\eta=0,$$

which when solved for $\eta$, becomes*:

$$\eta=\frac{a_2}{[1+a_1^2]^{\frac{3}{2}}}\xi^2+\cdots.$$

By the same transformation the equation

$$z-z_0=b_1(r-r_0)+b_2(r-r_0)^2+\cdots,$$

of the curve $c_2$, becomes

$$\eta=\frac{b_2}{[1+b_1^2]^{\frac{3}{2}}}\xi^2+\cdots.$$

From the relations given in § 6 we find†

$$\frac{1}{[1+a_1^2]^{\frac{3}{2}}}=\frac{1}{[1+b_1^2]^{\frac{3}{2}}}=-\frac{\left[\left(\frac{\partial f_1}{\partial r}\right)_0+\omega^2r_0\right]^3}{D^{\frac{3}{2}}},$$

where

$$D=\left[\left(\frac{\partial f_1}{\partial r}\right)_0+\omega^2r_0\right]^2+\left(\frac{\partial f_1}{\partial z}\right)_0^2=\left(\frac{\partial f_1}{\partial r}\right)_0^2+\left(\frac{\partial f_1}{\partial z}\right)_0^2+2\left(\frac{\partial f_1}{\partial r}\right)_0r_0\omega+r_0^2\omega^2.$$

Hence, if we denote the $\eta$'s of the curves $c_1$ and $c_2$ by $\eta_1$ and $\eta_2$, respectively, we obtain the equations

$$\eta_1=-\frac{1}{6}\frac{C+B_1\omega^2+A_1\omega^4}{D^{\frac{3}{2}}}\xi^2+\cdots,\qquad \eta_2=-\frac{C+B_2\omega^2+A_2\omega^4}{D^{\frac{3}{2}}}\xi^2+\cdots,$$

which are the equations (1) and (2) already given. As has been stated, the southerly deviation of a falling body is the difference $\eta_2-\eta_1$, when $\xi$ is replaced by the height $h$‡ through which the body falls, and $f_1$ is a sufficiently close approximation to the potential function of the earth's gravitational field of force.

* The equation

$$0=Ey+Ax^2+Bxy+Cy^2+Lx^3+Mx^2y+Nxy^2+\cdots$$

when solved for $y$ becomes

$$y=-\frac{A}{E}x^2+\cdots.$$

† The negative sign is taken because $[1+b_1^2]^{\frac{3}{2}}$ and $D^{\frac{3}{2}}$ are positive and $(\partial f_1/\partial r)_0+\omega^2r_0$ is negative.

‡ As already remarked, the circle of reference must be determined by a plumb-line of length $h$ supported at the initial point of the falling body.

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# AN APPLICATION OF A (1, 2) QUATERNARY CORRESPONDENCE TO THE WEDDLE AND KUMMER SURFACES*

BY

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*Introduction.*

The three-dimensional non-linear involution determined by a system of quadrics having three linear parameters has been extensively discussed. In the case in which all the quadrics have six basis points the system furnishes a convenient method of mapping the Weddle surface upon the Kummer surface. In the following paper I develop the details of this transformation analytically, obtaining a number of results already known, but by new methods, and apply the transformation to obtain a number of series of transformations which leave the surfaces invariant. A number of new transformations appear when the basis points are in simple or multiple involution. The relations which must exist among the coefficients of the equations to insure involution, the positions of the new lines lying on the Weddle surface, and the corresponding specializations of the Kummer surface are obtained.

## § 1. *The Involution I.*

1. Through six arbitrary points in space can be passed a linear system of quadric surfaces having three parameters. The equation of the system has the form

$$H \equiv \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 + \lambda_4 H_4 = 0. \tag{1}$$

If the coördinates $(\xi')$ of a point $P'$ are substituted in $H$, between $H(x)$ and $H(\xi')$ one parameter, as $\lambda_4$, may be eliminated, and thus the net

$$\lambda_1(H_1 \cdot H_4' - H_1' \cdot H_4) + \lambda_2(H_2 \cdot H_4' - H_2' \cdot H_4) + \lambda_3(H_3 \cdot H_4' - H_3' \cdot H_4) = 0$$

is obtained. The coefficients of the $\lambda_i$ define three quadrics which intersect in eight points; as seven of these points are known the coördinates of the eighth, $P'' \equiv (\xi'')$, can be expressed as rational functions of the coördinates of the six fixed points and $(\xi')$. In other words, the equations

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(2) $$\frac{H_1''}{H_1'} = \frac{H_2''}{H_2'} = \frac{H_3''}{H_3'} = \frac{H_4''}{H_4'}$$

yield the rational solutions

$$\xi_i'' = \phi_i(\xi_1', \xi_2', \xi_3', \xi_4').$$

*The seventh and eighth basis points of any net of quadrics contained in a linear system having six fixed basis points define an involutorial birational transformation.**

Without restricting the problem, the basis points may be taken at the vertices of the tetrahedron of reference, the unit point and the point $(a)$. It will be convenient to designate the points by number, as follows:

$$(1,0,0,0) \equiv 1, \quad (a_1a_2a_3a_4) \equiv 2, \quad (0,1,0,0) \equiv 3, \quad (1,1,1,1) \equiv 4,$$
$$(0,0,1,0) \equiv 5, \quad (0,0,0,1) \equiv 6.$$

In the net defined by $1, 2, \cdots, 6, (\xi'), \xi'')$, the line $(\xi'\xi'')$ determines a pencil of quadrics whose basis curve consists of this line and the cubic curve $\gamma_3$ passing through $1, 2, \cdots, 6$. The line $(\xi'\xi'')$ is a bisecant of the cubic curve.

*Any two associated points* $(\xi')$, $(\xi'')$ *lie on a bisecant of the cubic curve* $\gamma_3$ *defined by the six basis points.* There are fifteen special bundles, each defined by the points $1, 2, \cdots, 6$ and a line joining some two of them. These bundles will be designated by the symbols $\{ik\}$. Similarly, if $P'$ is on $\gamma_3$, the net will contain $\gamma_3$. It will be called $\{\gamma_3\}$.

2. A line $l$ (or any three points upon it) will uniquely define a quadric $H$ which contains it. When $(\xi')$ describes $l$, $(\xi'')$ will describe some curve upon $H_l$, always lying on the bisecant of $\gamma_3$ through $(\xi')$. The bisecants of $\gamma_3$ from points on $l$ define a ruled surface $R_4$ of order four, having $l$ for simple directrix and $\gamma_3$ for double curve. $R_4$ and $H_l$ intersect in a curve of order eight, composed of $l$ and a curve $c_7$, of order seven, the locus of $(\xi'')$. A plane through $l$ cuts $H_l$ in a straight line and cuts $R_4$ in three straight lines. The points of intersection are on $c_7$, hence $l$ cuts $c_7$ in four points. *By the involutorial transformation I straight lines go into space curves of order seven, each cutting its image line in four points. Planes go into surfaces of order seven.*

3. Since $R_4$ has $1, 2, \cdots, 6$ for double points and $H_l$ passes through each, $c_7$ has six double points which are principal points of the transformation, the image of each being a quadric cone having its vertex at its associated point and containing $\gamma_3$ and five lines $(ik)$. These cones, each counted twice, constitute

*GEISER: *Ueber zwei geometrische Probleme*, Crelle's Journal, vol. 67 (1867), pp. 78–95; EBERHARDT: *Ueber eine räumlich involutorische Verwandtschaft 7. Grades und ihre Kernfläche 4. Ordnung*, Breslau dissertation (1885), 61 pp.; STURM: *Die Lehre von den geometrischen Verwandtschaften*, vol. 4 (1909), pp. 414–418. For the literature on the system of surfaces see also STAUDE: Encyklopädie der mathematischen Wissenschaften, III C2, § 141 (p. 252).

the entire Jacobian of the system $\phi$. The curve $\gamma_3$ and the lines $(ik)$ are principal curves of the second kind, such that the image of a point on any one of them is the whole line on which it lies.

4. A plane $\pi$ goes into a surface $F_7$ of order seven. Since $\pi$ has three points on $\gamma_3$, $F_7$ has $\gamma_3$ as a threefold curve, and similarly, $F_7$ contains each of the lines $(ik)$. Two planes intersect in $l$; their image surfaces intersect in $c_7$ the image of $l$, in $\gamma_3$ counted nine times and in the fifteen lines $(ik)$. Hence there are no further principal points in the transformation.

5. If $l$ passes through a basis point $i$, the principal cone $K_2^{(i)}$ is part of its image. If $l$ is the intersection of two planes $\pi, \pi'$, each passing through $i$, the image $c_7$ will be the intersection of $F_7, F_7'$. Both these surfaces contain $k_2(i)$ as factor. Each of the residual quintics contains $\gamma_3$ as double curve and the ten lines $(lk)$ not passing through $i$. Thus the proper image of $l$ is a cubic curve passing through the basis points other than $i$, and having $l$ for bisecant.

## §2. *Locus of Coincident Points.*

6. If $P'' = P'$, all the surfaces of the associated net touch each other at $P'$, and one surface has a double point, or $P'$ is the vertex of a cone belonging to (1). The equation of the locus of $P'$ is expressed by the vanishing of the Jacobian of (1). It can be reduced to the form

$$S_4 \equiv |a_1 x_2 x_3 x_4 \; x_1 \; a_1 \; 1| = 0.$$

The locus is the Weddle quartic surface. It contains $1, 2, \cdots, 6$ as double points, the curve $\gamma_3$, the fifteen lines $(ik)$ and the ten lines of intersection of the planes $(ikl)$, $(mnp)$; these lines $(ikl)$, $(mnp)$ are defined by the symbol $(ikl)$. The four points of intersection of a line $l$ and its image $c_7$ lie on $S_4$, hence a general line $l$ contains no pair of associated points $(\xi')$, $(\xi'')$. But it was seen that the lines joining $(\xi')$ to $(\xi'')$ are bisecants of $\gamma_3$, hence each of these bisecants contains an entire involution of associated points. The points on $\gamma_3$ constitute a pair and the remaining intersections with $S_4$ are the double points of the involution. The Weddle surface therefore cuts harmonically the bisecants of the cubic curve $\gamma_3$ which lies upon it.

7. In the involution $T_3$ of conjugate points as to $\gamma_3$, $S_4$ is invariant. Moreover, $I$ can now be defined independently of (1) if the points $1, 2, \cdots, 6$ are given, since $S_4$ and $\gamma_3$ are fixed. Given a point $P'$ in space. Let the bisecant of $\gamma_3$ from $P'$ cut $S_4$ in $M_1, M_2$. The harmonic conjugate of $P'$ as to $M_1, M_2$ is $P''$. From the theorem of STEPHANOS* it follows that the transformations $T_3$ and $I$ are commutative.

* Mathematische Annalen, vol. 22 (1883), pp. 299–336. See page 313.

8. A plane $\pi$ cuts its image $F_7$ in a curve of order seven and $S_4$ in a quartic which lies on $F_7$. The residual intersection consists of the three bisecants of $\gamma_3$ in $\pi$. A line cutting $\gamma_3$ goes into $c_7$ consisting of $\gamma_3$ and a rational quartic passing simply through all basis points and cutting the given line at the three residual intersections with $S_4$.

## § 3. *Transformations of $S_4$ into Itself.*

9. We may now enumerate the involutorial birational transformations of $S_4$ into itself:

The six nodal projections $(N_1)$;

The transformation $T_3$, defined by conjugate points as to $\gamma_3$;

The fifteen transformations defined by a secant of $\gamma_3$ and one of its bisecants $(ik)$. Given any point $P$ on $S_4$. The cubic cone projecting $\gamma_3$ from $P$ will be cut by the plane $P,(ik)$ in the two lines $P_i$ and $P_k$ and one other line. If the latter cuts $S_4$ in $Q$, between $P$ and $Q$ exists a birational correspondence. It will be designated by $T_{\gamma ik}$;

The interchange of points on every secant of two skew lines $T_{ik,\,lm}$ or $T_{ik,\,lmn}$;

The six transformations defined by the cubic curves passing through a point $P$ and five basis points. The residual point of intersection of the curve with $S_4$ is fixed when $P$ is known. We shall call these transformations $C_i$, $i$ being the symbol of the basis point through which the curve does not pass;

The transformations similar to the last preceding ones defined by the cubic curves passing through four basis points and having the line joining the remaining pair for bisecant. These transformations will be designated by $C_{ik}$;

An infinite number of further involutions can be defined as follows. Through any point on $\gamma_3$, a point of any fundamental line $(ik)$ and a point $D$ on $S_4$ not on any fundamental line pass a quadric surface. It will belong to (1), and cut $S_4$ in $\gamma_3$, $(ik)$ and a space quartic passing through the remaining nodes. To get the point associated with any point $P$ on $S_4$, pass a plane through $P$ and any two nodes $l$, $m$ other than $i$, $k$. The plane cuts the space quartic in $l$, $m$ and two other points. Pass a conic through these four points on the quartic and through $P$. It is fixed, and will cut $S_4$ in one remaining point $Q$. The conjugate of this conic in $I$ is a sextic curve having double points at the basis points through which the conic does not pass, and passing simply through the six points in which the conic cuts $S_4$. The operation of interchanging $P$, $Q$ will be denoted by $X_D$. The same

series of curves is invariant under $x$ as under $(N_i, N_m)$, but the former can be expressed in terms of the latter only when the line joining the points on the quartic is tangent to the curve elsewhere. But with the exception of the points $i$, $k$ this is impossible, since one point lies on $\gamma_3$ and the other on $(ik)$. The group generated by $X_D$, $X_{D'}$, is of infinite order.

10. The points of $\gamma_3$ are defined in terms of the homogeneous parameters $\sigma$, $\tau$ by the equations

$$\rho x_1 = A\sigma\tau(\sigma - \tau),$$
$$\rho x_2 = B\sigma(\sigma - \tau)(\sigma_1\tau - \sigma\tau_1),$$
$$\rho x_3 = C\sigma\tau(\sigma_1\tau - \sigma\tau_1),$$
$$\rho x_4 = D\tau(\sigma - \tau)(\sigma_1\tau - \sigma\tau_1),$$

wherein

$$A = a_1(a_2 - a_3)(a_2 - a_4)(a_4 - a_3), \quad B = a_2(a_4 - a_3), \quad C = a_3(a_2 - a_4),$$
$$D = a_4(a_2 - a_3), \quad \sigma_1 = (a_2 - a_3)(a_1 - a_4), \quad \tau_1 = (a_1 - a_2)(a_4 - a_3).$$

The values of the parameters at the basis points are:

| | | |
|---|---|---|
| at 1, | $\sigma = (a_2 - a_3)(a_1 - a_4)$, | $\tau = (a_1 - a_2)(a_4 - a_3)$, |
| at 2, | $\sigma = a_2 - a_3$, | $\tau = a_4 - a_3$, |
| at 3, | $\sigma = 1$, | $\tau = 0$, |
| at 4, | $\sigma = a_4(a_2 - a_3)$, | $\tau = a_2(a_4 - a_3)$, |
| at 5, | $\sigma = 1$, | $\tau = 1$, |
| at 6, | $\sigma = 0$, | $\tau = 1$. |

These equations may be used in $C_2$ by replacing $(a)$ by the coördinates of $P$. The other $C_i$ can be obtained from it by linear transformations.

The equations of the Cremona transformation $T_3$ can be obtained from the preceding equations alone.

## § 4. *Special Positions of Basis Points.*

11. The quadric surface determined by the lines (12), (35), (46) has the equation

$$x_4[x_1(a_2 - a_3) + a_3x_2 - a_2x_3] + x_1[a_4x_3 - a_4x_2] = 0.$$

It cuts $\gamma_3$ in the six basis points; if upon the coefficients the condition is

imposed that one further point of $\gamma_3$ lies on the quadric, then $\gamma_3$ lies entirely upon it. This condition is

$$a_2(a_1 - a_3) = a_1(a_4 - a_3).$$

In this case the four nets $\{\gamma_3\}$, $\{12\}$, $\{35\}$, $\{46\}$ have a common quadric.

The intersection of a quadric with $S_4$ consists of a curve of order eight cutting each generator in both reguli in four points. On one regulus the points of intersection are on the three lines (12), (35), (46) and on $\gamma_3$; on the other, two of them are on $\gamma_3$, hence the residual intersection consists of a curve of order two cutting the generators of the second regulus in two points and not cutting those of the first, that is, it is composed of two straight lines $u$, $v$.

The equations of these lines are

$$lx_1 - mx_4 = 0, \qquad x_1(a_2 l - a_3 l) + x_2(a_3 l - a_4 m) + x_3(a_4 m - a_2 l) = 0,$$

the parameters $l : m$ being defined by the quadratic equation

$$a_1 l^2 - 2a_1 l_m + a_4 m^2 = 0.$$

If the points $1, 2, \cdots, 6$ are projected from any point of $\gamma_3$ upon a plane, the generator of the same regulus as $u$, $v$ which passes through the center of projection will pierce the plane of projection in a point through which pass the projections of (12), (34), (56). The basis points thus project into six points in involution. Conversely, when from any point of $\gamma_3$ these points are projected into points in involution, they and $\gamma_3$ have a common transversal and the preceding condition applies. We shall speak of the six points $1, 2, \cdots, 6$ themselves as being in involution when their projections are in involution.

*When the six basis points* $1, 2, \cdots, 6$ *are in involution the surface* $S_4$ *contains two new lines* $u$, $v$. The generators of the regulus cutting $u$, $v$ are bisecants of $\gamma_3$, cutting $S_4$ in the residual points on $u$, $v$. These four points of intersection are harmonic. In the axial involution determined by $u$, $v$ the six points $1, 2, \cdots, 6$ are interchanged in pairs and $\gamma_3$ goes into itself, hence $S_4$ must remain invariant. *In this case the axial involution* $I_1$ *must be added to the preceding transformations.*

12. In the same manner it is found that (13), (24), (56) are in involution if the $a_i$ satisfy the condition $a_1 a_2 - a_3 a_4 = 0$;

(16), (34), (25) are in involution if $a_1(a_3 - a_4) = a_3(a_2 - a_4)$;

(12), (36), (45) " " " " $a_1(a_3 - a_4) = a_2(a_1 - a_4)$;

(14), (23), (56) " " " " $a_2(a_4 - a_1) = a_3(a_4 - a_2)$;

(15), (26), (34) " " " " $a_1(a_4 - a_3) = a_4(a_2 - a_3)$.

These six conditions are all satisfied when the $a_i$ have the values

$$a_1 = 1, \qquad a_2 = 2, \qquad a_3 = 1 + i, \qquad a_4 = 1 - i.$$

*The surface $S_4$ now has six pairs of lines $u_i$, $v_i$ and is invariant under six axial involutions $I_i$.* No two of these involutions can be commutative. In order that two axial involutions may be commutative it is necessary that their axes shall either intersect or belong to the same regulus. Let $(u, v)$, $(u', v')$ be the axes and suppose $u$, $u'$ intersect in $M$. The plane of $u$, $u'$ cuts $v$ in $K$, $v'$ in $L$. Any line through $K$ cuts $S_4$ in a fourth point which is the harmonic conjugate of its intersection with $u'$ as to its intersection with $u$ and $K$. The locus of this point is a line through $M$. The residual intersection of the plane consists of another line, this latter passing through $K$. But this is impossible, as $S_4$ cannot contain so many coplanar lines.

If $(u, v)$, $(u', v')$ belong to the same regulus, they must be harmonic, since any transversal cuts $S_4$ only on these lines. The quadric determined by them would cut $S_4$ in four lines belonging to the other regulus, of which one at most can be a line $(ik)$. Of the six lines $(ilm)$ skew to $(ik)$, three can be chosen that are mutually skew and do not have another fundamental line for common transversal. Since $u$, $v$, $u'$, $v'$ are all invariant under $I$, a generator of the same regulus goes into a sextic cutting every generator of each system in three points. But this curve has double points at the fundamental points except $i$ and $k$, hence all the fundamental points lie on the quadric, which therefore belongs to the system (1). Since $(lm)$ cuts $u$, $v$, it must therefore lie on the quadric, but this is impossible, as the entire intersection with $S_4$ has been accounted for. Hence in this case $u$, $v$, $u'$, $v'$ cannot belong to the same regulus. Similarly for the case when no line $(ik)$ cuts both pairs of axes. This completes the proof that no two operations $I_i$, $I_k$ are commutative.

## § 5. *Double λ-Space.*

In (1) let the $\lambda_i$ be considered as homogeneous plane coördinates. The plane will define a quadric of the system (1). A linear equation will define a point in $(\lambda)$. This equation and (1) will determine a net of quadrics, passing through $1, 2, \cdots, 6$ and two associated points $(\xi')$, $(\xi'')$. If the coördinates of $(\xi')$ are substituted in (1), the resulting equation will fix the same point in $(\lambda)$. Thus, the equation (1) defines a (1, 2) correspondence between the points $(\lambda)$ of a space $\Sigma'$ and the points $(\xi)$, $(\xi'')$ of another space $\Sigma$. Two linear equations in $(\lambda)$ define a straight line in $\Sigma'$. From these equations and (1) a pencil of quadrics is fixed, whose basis curve passes through the basis points $1, 2, \cdots, 6$. Hence when $(\xi')$ describe a plane in $\Sigma$, $(\lambda)$ will describe a quartic surface. The complete image in $\Sigma$ of this quartic surface in $\Sigma'$ is the plane and its image $F_7$

in $I$. To a straight line in $\Sigma$ corresponds a conic in $\Sigma'$, whose complete image in $\Sigma$ is the line and its image $c_3$.

If in the first derivatives of (1) we eliminate $x_i$, the resulting equation of order four in $\lambda_i$ will define those values for which $(\xi') \equiv (\xi'')$. They define a surface of class four, $\Delta(\lambda)$. If $\Sigma'$ is regarded as a double space, one sheet corresponding to $(\xi')$, the other to $(\xi'')$, the points of $\Delta(\lambda)$ are branch points. *The points of* $\Delta(\lambda)$ *and the points of* $S_4$ *are in* (1, 1) *correspondence.* Since an arbitrary line in $\Sigma$ is not its own image in $I$, the image conics cannot intersect $\Delta(\lambda)$, hence:

*The images of the* $\infty^4$ *lines of* $\Sigma$ *are* $\infty^4$ *conics which touch four tangent planes of* $\Delta$.

The image of a general plane $\pi$ touches $\Delta$ along a curve of order eight. To a plane section of $\Delta$ corresponds a curve of order eight, in which the image quadric cuts $S_4$. A tangent plane of $\Delta$ corresponds to a cone of $\Sigma$, belonging to (1). To the point of contact corresponds its vertex. To the four cones in a pencil of quadrics belonging to (1) correspond the four tangent planes through the image line of the basis quartic.

14. To an arbitrary point $\Sigma a_i \lambda_i$ corresponds a pair of points $(\xi')$, $(\xi'')$. The $\infty^2$ planes of $\Sigma'$ through this point correspond to the $\infty^2$ quadrics through $1, 2, \cdots, 6, (\xi'), (\xi'')$. In particular, to the tangent planes to $\Delta$ passing through the point correspond $\infty^1$ cones in the net of quadrics. The locus of the vertices of these cones is a curve of order eight on $S_4$, having double points at all eight basis points. If in $\Sigma$ we project the $\infty^2$ basis curves of order four and this octic from one of the basis points, and take the dual of the configuration in $\Sigma'$ as to a quadric surface, we have exactly the Geiser transformation which was discussed in a preceding paper.*

Let $P'_{00}$ in $\Sigma'$ correspond to the net of quadrics $\{\gamma_3\}$. It is a principal point of $\Sigma'$. Similarly, let $P'_{ik}$ correspond to $\{ik\}$. It is also a principal point. Any plane $\pi$ in $\Sigma$ cuts $\gamma_3$ in three points and each of the fifteen lines $(ik)$ once. Its image $F'$ in $\Sigma'$ passes three times through $P'_{00}$ and once through each of the fifteen points $P'_{ik}$. Since $\gamma_3$ and $(ik)$ all lie on $S_4$, $P'_{00}$, $P'_{ik}$ all lie on $\Delta$. The $c'_8$ of contact of $\Delta$ and $F'_4$ passes three times through $P'_{00}$ and simply through $P'_{ik}$.

Every bisecant of $\gamma_3$ contains a whole involution of points $(\xi')$, $(\xi'')$. Its image in $\Sigma'$ is a straight line through $P'_{00}$, counted twice. *The bundle of lines through* $P'_{00}$ *in* $\Sigma'$ *and the* (1, 3) *congruence of bisecants of* $\gamma_3$ *are in* (1, 1) *correspondence.*

Since $\pi$ cuts $\gamma_3$ in three points and contains three bisecants, $F'_4$ contains three double lines passing through $P'_{00}$. Thus $F'_4$ is a Steiner surface. A line $l$ in $\pi$ corresponds to a conic $c'_2$ on $F'_4$. The plane $\omega'$ of $c'_2$ corresponds to a

* SNYDER: *Conjugate line congruences contained in a bundle of quadric surfaces*, these Transactions, vol. 11 (1910), pp. 371–387.

quadric $H$ containing $l$ and another line $m$ lying in $\pi$; the image of $m$ is a second conic $k_2'$ in $\omega'$ on $F_4'$. These two conics intersect on the three double lines and in one other point, the image of $lm$. $H$ touches $\pi$ in $lm$; $\omega'$ touches $F_4'$ in $c_2'$, $k_2'$. Both of these conics are contact curves of $\Delta$.

15. The points $1, 2, \cdots, 6$ are principal points of the transformation; the image of each is a plane $\pi_i'$. All the basis points lie on $\gamma_3$, hence the six planes $\pi_i'$ pass through $P_{00}'$; $(ik)$ passes through two, hence $\pi_i'$, $\pi_k'$ pass through $P_{ik}'$. The fifteen lines $P_{00}'P_{ik}'$ are therefore the intersections of the six planes $\pi_i'$, $\pi_k'$. Since a plane in $\Sigma'$ is the image of a quadric in $\Sigma$, the planes $\pi_i'$ may be regarded as images of the cones in (1), having $i$ for vertex and passing through the other basis points. A bisecant of $\gamma_3$ cuts $S_4$ in two points not on $\gamma_3$; its image cuts $\Delta$ in two points not at $P_{00}'$.

A basis $c_4$ in $\Sigma$ cuts $S_4$ in four points besides the basis points, hence its image line cuts $\Delta$ in four points. *$\Delta$ is of order four. The surface of branch points in $\Sigma'$ is of order and class four and has sixteen double points $P_{00}'$, $P_{ik}'$, hence it is a Kummer surface.* To a straight $l'$ through $P_{ik}'$ corresponds a quartic curve in $\Sigma$ consisting of $(ik)$ and a cubic curve cutting $(ik)$ twice and passing through the other four basis points. The curve cuts $S_4$ in two points not at basis points nor on $(ik)$, hence its image line cuts $\Delta$ in two points not at $P_{ik}'$, or $P_{ik}'$ is also a double point.

16. Any line $l'$ in $\pi_i'$ goes into a $c_4$ on $K_2^{(i)}$, but as this fundamental cone is the tangent cone to $S_4$ at $i$, $c_4$ has a double point at its vertex, and all the variable points of intersection with $c_4$, $S_4$ are at $i$, two on each nodal tangent. Thus $l'$ is a double tangent to $\Delta$, or $\pi_i'$ contains a conic of contact. It passes through $P_{00}'$ and $P_{ik}'$, $(k = 1, 2, \cdots, 6, i \neq k)$.

The plane $(123)$ goes into an $F_4'$ consisting of $\pi_1'$, $\pi_2'$, $\pi_3'$ and another plane $\pi_{123}'$, whose complete image in $\Sigma$ consists of $(123)$ and its image in $I$, viz., $(456)$. The line $(123), (456)$ goes into the conic of contact of $\pi_{123}'$. The principal points lying upon it are $P_{12}'$, $P_{13}'$, $P_{23}'$, $P_{45}'$, $P_{46}'$, $P_{56}'$. The distribution of singular points and singular planes can now be readily determined by observing the corresponding elements in $\Sigma$.

A line $l$ through any basis point $i$ in $\Sigma$ corresponds to a line in $\Sigma'$, for, if any quadric $H$ of (1) corresponds to the plane $p'$, $l$ cuts $H$ in as many variable points and the image cuts $p'$. But $H$ passes through $i$, hence $l$ corresponds to a straight line $l'$. Since $l$ does not correspond to itself in $I$, $l'$ is a bitangent of $\Delta$. Thus the bitangents of $\Delta$ are arranged in six distinct systems corresponding to the six basis points $1, 2, \cdots, 6$, in $\Sigma$.

## § 6. *Image Congruences of the Bundles* $1, 2, \cdots, 6$.

17. Through any point $P'$ in $\Sigma'$ pass two lines of the congruence $[i]$, since $P'$ has two image points $(\xi')$, $(\xi'')$ in $\Sigma$, each of which defines a line of $i$. A

quadric $H$ of (1) contains two lines of $i$, hence its image plane contains two lines of $[i]$. Any line $l$ of $i$ cuts $S_4$ in two residual points. These go into the points of contact of $l'$ with $\Delta$. The complete image in $\Sigma$ of $l'$ consists of $l$ and its image in $I$, that is, a space cubic curve passing through the remaining basis points and cutting $l$ at its residual intersection with $S_4$.

The arrangement of the lines of these congruences can now be ascertained. The bundle $i$ can be considered as made up of a single infinity of plane pencils each in a plane through $(ik)$. The image of each plane is a Steiner surface $F'_4$ consisting of $\pi'_i$ and $\pi'_k$ and a quadric surface. The generators of one regulus are the images of the lines of the pencil $i$, and those of the other regulus are images of the lines through $k$. These quadrics all pass through eight basis points and form a singly infinite quadratic system. Their envelope is the surface $\Delta$.*

## § 7. *Depiction of $S_4$ on $\Delta$.*

18. We can now study the transformations which leave $\Delta$ invariant by depicting $S_4$ upon $\Delta$ and considering the transformations into which those enumerated in No. 9 are transformed. From the preceding section it follows immediately that the nodal projections $(N_i)$ become those of interchanging the points of contact on the lines of congruences $[i]$.†

The six operations $[i]$ generate an abelian group of order 32. The product of any two of them is an axial homology. By No. 14 the operation $T_3$ of conjugate points as to $\gamma_3$ becomes the nodal projection $(P'_{00})$ of $\Delta$ into itself from $P'_{00}$. The involution $I$ is transformed into a Jonquières transformation leaving every point of $\Delta$ invariant. To find the image of any point $A'$, connect it with $P'_{00}$. The line $A'P'_{00}$ cuts $\Delta$ in $M'_1$, $M'_2$. The harmonic conjugate of $A'$ as to $M'_1 M'_2$ is the required image.

19. In $\Sigma'$, the nodal projection $(P'_{ik})$ can be obtained by transforming $(P'_{00})$ through $[i][k]$,

$$(P'_{ik}) = [i][k](P'_{00})[k][i].$$

In $\Sigma$ this operation requires that a point be projected from $i$, its image from $k$, this image found as to $\gamma_3$, then the first two operations repeated in reverse order Pass two quadrics $H_1$, $H_2$ of the system (1) through the line $(ik)$. Their

* The equations of these quadrics, the configuration of the associated basis points, and all the properties of the congruences of order and class two are at once derived by projecting the cubic variety having 10 nodes in four way space from a point upon it, as is outlined in my paper: *Surfaces derived from the cubic variety having nine double points in four dimensional space*, these Transactions, vol. 10 (1909), pp. 71-78. In the notation of that paper, $\lambda = 1$ will define as contour a general Kummer surface.

† This result was first found by H. F. Baker by another method. A different proof is given in my paper: *Infinite discontinuous groups of birational transformations which leave certain surfaces invariant*, these Transactions, vol. 11 (1910), pp. 15-24. See page 17 for further references.

residual intersection consists of a cubic passing through the other basis points, and cutting $(ik)$ twice. The cubic is fixed by any point $P$ on $S_4$. The images in $\Sigma'$ of $H_1$, $H_2$ are two planes each passing through $P'_{ik}$. Their intersection consists of a straight line which is fixed by $P'$, the image of $P$.

*The fifteen operations $C_{ik}$ are transformed into the fifteen nodal projections $(P'_{ik})$.*

$$C_{ik} = (N_i)(N_k)T_3(N_k)(N_i).$$

20. Any two of these operations generate a group of infinite order. Moreover, the index of this group as to that generated by two pairs will also be infinite. This can be most easily proved in $\Sigma'$. Let $(P'_{12})(P'_{34}) = s$ and $(P'_{15})(P'_{36}) = t$.

Pass a plane through $P'_{12}P'_{34}$ and any point $P'$ on $\Delta$. This plane cuts $\Delta$ in a general quartic curve having two nodes. When $P'$ is operated upon by $s$, its image will always lie on the same quartic curve. Pass a plane through $P'$, $P'_{15}$, $P'_{36}$. It will cut $\Delta$ in a quartic curve which intersects the preceding one in three points $A'_1$, $A'_2$, $A'_3$ besides $P'$. The images of $P'$ when operated upon by $t$ will lie in the second plane. If any power of $s$ were equal to any power of $t$, the corresponding position of $P'$ would be at $A'_i$, hence these respective operations repeated four times would cause the point to return to $P'$. If the period is finite for one point it has been proved to be finite for every point, but $s$ and $t$ are each of infinite order. Hence no relation of the form $s^m = t^n$ can be found.

Now multiply the series $1, s, s^2, s^3 \cdots$ by $t, t^2, \cdots$ on the right. If $s^k t^l = s^p t^q$, then $s^{k-p} = t^{q-l}$, which relation was proved to be impossible. It follows that no operation $s^k t^l$ appears twice in this set, or that the index of the group generated by $s$ and $t$ is of infinite order as to the group generated by $s$. In the same manner we can obtain a group generated by $s, t, u$, having an infinite index as to that generated by $s, t$, etc. All the operations of the group can be expressed in terms of $(P'_{00})$ and the operators of the linear group $G_{16}$.

21. Since the operations $C_i$ and $(N_i)$ are transformed into each other by $I$, it follows that $C_i$ goes into $[i]$ in $\Sigma'$.

22. The straight lines defining $T_{\gamma ik}$ go into conics in $\Sigma'$ passing through $P'_{00}$ and $P'_{ik}$. Since the lines in $\Sigma$ do not go into themselves when operated upon by $I$, it follows that the image conics in $\Sigma'$ touch $\Delta$ in the two points which are images of the residual points of intersection of the line and $S_4$. Through any point $P$ of $S_4$ can be drawn but one line cutting $\gamma$ and $(ik)$; it will cut $S_4$ in $Q$. The basis curve composed of $\gamma$ and $(ik)$ will define a pencil of quadric surfaces, one of which passes through $P$. This quadric goes into the plane passing through $P'_{00}$ and $P'_{ik}$, cutting $\Delta$ in a binodal quartic curve. Through $P'$ can be passed four conics, all of which touch $\Delta$ at $P'$, pass through $P'_{00}$, $P'_{ik}$, and touch $\Delta$ in another point. The generators of the other regulus on the quadric in $\Sigma$ go into $\infty^1$ conics which are proper contact conics, that do not pass through

the nodes. Any binodal quartic can be generated in four ways as the envelope of a conic passing through the nodes; its pole as to the line joining them describes a conic. The four loci of poles are inscribed in the same quadrilateral, having a vertex at each node. The series of conics are rationally separated when one position of the pole is known. The second point of contact of one conic is $Q'$. When $P'$ describes $\Delta$, $Q'$ can be distinguished from the other points of contact by means of the equation of the locus of the pole. Each of the other series of contact conics will also define birational transformations, but these are not images of straight lines in $\Sigma$. The conjugate operation in $I$ is defined by the cubics passing through four basis points and cutting $\gamma$ in one variable point.

In the same way, the operations $T_{ik,\,lm}$ are transformed into the operations defined by the series of contact conics passing through $P'_{ik}$, $P'_{lm}$. They are obtained from the preceding series by the linear transformations $[i]\,[k]$. Now consider the lines (123), (14) in $\Sigma$. The former goes into a conic through $P'_{12}$, $P'_{13}$, $P'_{23}$, $P'_{45}$, $P'_{46}$, $P'_{56}$, or the singular conic in the plane $\pi'_{123}$. The line (14) goes into the point $P'_{14}$. Any line in $\Sigma$ cutting (123), (14) goes into a conic in $\Sigma'$ passing through $P'_{14}$, touching the conic of contact in the singular plane $\pi'_{123}$, and touching $\Delta$ in the images of the residual intersections of the line and $S_4$. As before, three extraneous series of conics appear.

23. The conjugate of the lines cutting $(ik)$, $(lm)$ in $I$ are quintic curves through $i$, $k$, $l$, $m$ simply, through the remaining basis points twice, having one variable point on $(ik)$ and on $(il)$. They are uniquely fixed by a point $P$ on $S_4$ and pass through $Q$, the residual point of intersection of the line through $P$ cutting $(ik)$, $(lm)$. As in the case of the cubics $C_i$, these curves could be used to define the transformation. Whenever the corresponding transformation in $\Sigma'$ is defined by rational contact curves, two distinct systems of rational curves in $\Sigma$ will define the same transformation of points on $S_4$. On the other hand a Cremona transformation in $\Sigma'$ may not be expressible in rational form in $\Sigma$, except for points of $S_4$.

24. To obtain the operation into which $X$ is transformed in $\Sigma'$, it will be observed that the quadric passing through $\gamma_3$ and $(ik)$ goes into a plane, and the section of the former with $S_4$ becomes a plane quartic curve, section of $\Delta$, having nodes at $P'_{00}$ and $P'_{ik}$. This section is fixed when one point upon $\Delta$ is given.

Through the conic passing through $l$, $m$ can be passed a quadric belonging to (1), hence its image in $\Sigma'$ is a plane. The plane of the conic is transformed into a Steiner surface having $\pi'_l$, $\pi'_m$ as factors, hence a quadric. The generators of one regulus are the images of the pencil of lines through $l$ in the plane of the conic, and those of the other are images of the lines through $m$. The image of the conic is therefore a conic. On $\Sigma$, the space quartic passes through all the basis points except $i$ and $k$. A plane through any other two will cut the curve in but two other points. The image quadric will therefore have but two points in common

with the image plane curve, that is, the conic in the plane of section will touch the curve at the two points. Through any point in the section of the quadric with $\Delta$ can be drawn a conic touching the curve which its plane cuts from $\Delta$ at two fixed points, the chosen point $P'$ and at one other point $Q'$. This is the transform of the operation $X$.

## § 8. *The Tetrahedroid.*

25. When the points (1, 2), (3, 5), (4, 6) are in involution, it was seen that the nets $\{\gamma_3\}$, $\{12\}$, $\{35\}$, $\{46\}$ have a quadric in common. This quadric becomes a plane in $\Sigma'$ passing through the four points $P'_{00}$, $P'_{12}$, $P'_{35}$, $P'_{46}$. Since the section of $\Delta$ made by the plane is a quartic having four double points, the curve of section consists of two conics. By the sixteen linear operations which leave $\Delta$ invariant the plane of the four nodes can be transformed into three others, forming with the given one a tetrahedron. By the operations $[i]$, the dual configuration is obtained, such that four points exist through each of which pass four of the singular planes of $\Delta$. These planes intersect in six lines, on each of which lie two nodes, hence the nodes on any singular conic are in involution.

*If the six nodes of $S_4$ are in involution, the six nodes in every singular conic of $\Delta$ are in involution. $S_4$ now has an extra pair of lines and $\Delta$ has four pairs of conics.*

26. The axial involution in $\Sigma$ goes into the operation defined by interchanging the points of contact of a conic which touches $\Delta$ in the images of the axes of the involution and in two other points. The four points of contact are harmonic. $\Delta$ is now a tetrahedroid; since the lines joining two nodes define an involution in each singular plane, and four singular planes pass through the center of this involution, $\Delta$ is now invariant under the central involutions defined by the vertices and opposite faces of the tetrahedron formed by the planes containing pairs of conics on $\Delta$. These operations cannot be defined in terms of the axial involutions in $\Sigma$. Let $P'$ be the center and $\pi'$ the invariant plane. The image of $P'$ in $\Sigma$ is a pair of points $\xi', \xi''$ not lying on $S_4$; the image of $\pi'$ is the quadric containing $u, v$. A line through $P'$ cuts $\pi'$ in $\Pi'$ and $\Delta$ in two pairs of points $M'_i$, each harmonic with $P'\Pi'$. The image of the line is a quartic curve passing through the eight associated points $1, 2, \cdots, 6, \xi', \xi''$. It cuts the quadric in two points $\eta', \eta''$ and $S_4$ in four points, images of $M'_i$.

27. When the points $1, \cdots, 6$ are in multiple involution, $\Delta$ is a multiple tetrahedroid. In the case of six fold involution the new central involutorial perspective transformations and the linear $G_{16}$ generate a group of order $24 \cdot 16$.*

*SEGRE, *Sur un cas particulier de la surface de Kummer*, Leipziger Berichte, vol. 36 (1884), pp. 132-135.

# ON SEMI-DISCRIMINANTS OF TERNARY FORMS*

BY

O. E. GLENN

## § 1. *Introduction.*

It is well known that the number of independent conditional relations which must exist among the coefficients of a ternary form of order $m$ in order that it should be factorable into linear factors, distinct term for term, is $\frac{1}{2}m(m-1)$. Several writers,† among them BRILL, and GORDAN, have published methods for the determination of such sets of relations. Their results are, as a rule, expressed in the form of a covariant, the identical vanishing of which gives necessary and sufficient conditions for the factorability.

These methods are somewhat indirect, and from certain standpoints are unsatisfactory for the additional reason that the set of conditions given by the identical vanishing of such a covariant is always redundant.

Our aim in this paper has been to develop a direct method of attacking this problem. Our method leads to a set of conditional relations containing the exact minimum number $\frac{1}{2}m(m-1)$; that is, it leads to a set of $\frac{1}{2}m(m-1)$ independent seminvariants of the form, whose simultaneous vanishing gives necessary and sufficient conditions for the factorability. We shall call these seminvariants *semi-discriminants*‡ of the form. They are all of the same degree $2m-1$; and are readily formed for any order $m$ as simultaneous invariants of a certain set of *binary* quantics related to the original ternary form.

If a polynomial, $f_{3m}$, of order $m$, and homogeneous in three variables $(x_1, x_2, x_3)$ is factorable into linear factors, its terms in $(x_1, x_2)$ must furnish the $(x_1, x_2)$ terms of those factors. Call these terms collectively $a_{0x}^m$, and the terms linear in $x_3$ collectively $x_3 a_{1x}^{m-1}$. Then if the factors of the former were known, and were distinct, say

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† THAER, "*Ueber die Zerlegbarkeit einer ebenen Linie dritter Ordnung in gerade Linien,*" Mathematische Annalen, vol. 14 (1879); GORDAN, "*Das Zerfallen der Curven in gerade Linien,*" Mathematische Annalen, vol. 45 (1894); BRILL, "*Ueber die Zerfällung der Ternärformen in lineare Factoren,*" Mathematische Annalen, vol. 50 (1898).

‡ It seems desirable to introduce a new term for the members of such a minimum set of functions of the form's coefficients. Since the name discriminant is already in common use, and since discriminants are *invariants*, it seems natural to adopt here the term *semi-discriminant*.

$$a_{0x}^m \equiv a_{00}\prod_{i=1}^{m}(r_2^{(i)}x_1 - r_1^{(i)}x_2) \div \prod_{i=1}^{m}(r_2^{(i)}),$$

the second would give by rational means the terms in $x_3$ required to complete the several factors. For we could find rationally the numerators of the partial fractions in the decomposition of $a_{1x}^{m-1}/a_{0x}^m$, viz.:

$$\frac{a_{1x}^{m-1}}{a_{0x}^m} \equiv \frac{\prod_{i=1}^{m} r_2^{(i)}}{a_{00}} \sum_{i=1}^{m} \frac{\alpha_i}{r_2^{(i)}x_1 - r_1^{(i)}x_2},$$

and the factors of the complete form will be, of course,

$$r_2^{(i)}x_1 - r_1^{(i)}x_2 + \alpha_i x_3 \qquad (i=1, 2, \cdots, m).$$

Further, the coefficients of all other terms in $f_{3m}$ are rational integral functions of the $r^{(i)}$ on the one hand, and of the $\alpha_i$ on the other — symmetrical in the sets $(r_2^{(i)}, -r_1^{(i)}, \alpha_i)$. We shall show in general that all these coefficients in the case of any linearly factorable form are *rationally* expressible in terms of those occurring in $a_{0x}^m$, $a_{1x}^{m-1}$. Hence will follow the important theorem (§ 5):

**Theorem 1.** *If a ternary form $f_{3m}$ is decomposable into linear factors, all its coefficients, after certain $2m$, are expressible rationally in terms of those $2m$ coefficients. That is, in the space whose coördinates are all the coefficients of ternary forms of order $m$, the forms composed of linear factors fill a rational spread of $2m$ dimensions.*

We shall thus obtain the explicit form of the general ternary quantic which is factorable into linear factors (§ 5). Moreover, in case $f_{3m}$ is not factorable a similar development will give the theorem (§ 3),

**Theorem 2.** *Every ternary form $f_{3m}$, for which the discriminant $D$ of $a_{0x}^m$ does not vanish, can be expressed as the sum of the product of $m$ distinct linear forms, plus the square of an arbitrarily chosen linear form, multiplied by a "satellite" form of order $m-2$ whose coefficients are, except for the factor $D^{-1}$, integral rational seminvariants of the original form $f_{3m}$.*

## § 2. *A class of ternary seminvariants.*

Let us write the general ternary quantic in homogeneous variables as follows:

$$f_{3m} = a_{0x}^m + a_{1x}^{m-1}x_3 + a_{2x}^{m-2}x_3^2 + \cdots + a_{m0}x_3^m,$$

where

$$a_{ix}^{m-i} \equiv a_{i0}x_1^{m-i} + a_{i1}x_1^{m-i-1}x_2 + a_{i2}x_1^{m-i-2}x_2^2 + \cdots + a_{im-i}x_2^{m-i} \quad (i=1, 2, \cdots, m).$$

Then write

$$\frac{a_{1x}^{m-1}}{a_{0x}^m} = \frac{a_{1x}^{m-1}}{\prod_{k=1}^{m}(r_2^{(k)}x_1 - r_1^{(k)}x_2)} = \sum_{k=1}^{m}\frac{\alpha_k}{r_2^{(k)}x_1 - r_1^{(k)}x_2} \quad (a_{00} = r_2^{(1)}r_2^{(2)}\cdots r_2^{(m)});$$

and we have in consequence, assuming that $D \neq 0$, and writing

$$a'^{m}_{0r^{(k)}} = \left[\frac{\partial a^{m}_{0x}}{\partial x_1}\right]_{x_1=r_1^{(k)},\ x_2=r_2^{(k)}}, \qquad a''^{m}_{0r^{(k)}} = \left[\frac{\partial a^{m}_{0x}}{\partial x_2}\right]_{x_1=r_1^{(k)},\ x_2=r_2^{(k)}},$$

the results

(1) $$\alpha_k = r_2^{(k)} a^{m-1}_{1r^{(k)}} / a'^{m}_{0r^{(k)}} = -\, r_1^{(k)} a^{m-1}_{1r^{(k)}} / a''^{m}_{0r^{(k)}}.$$

Hence also

(2) $$a''^{m}_{0r^{(k)}} = -\frac{r_1^{(k)}}{r_2^{(k)}} a'^{m}_{0r^{(k)}}.$$

The discriminant of $a^{m}_{0x}$ can be expressed in the following form:

(3) $$D = \prod_{j=1}^{m} a'^{m}_{0r^{(j)}} / a_{00}(-1)^{\frac{1}{2}m(m-1)},$$

and therefore

(4) $$\alpha_k = \frac{r_2^{(k)} a^{m-1}_{1r^{(k)}} a'^{m}_{0r^{(1)}} a'^{m}_{0r^{(2)}} \cdots a'^{m}_{0r^{(k-1)}} a'^{m}_{0r^{(k+1)}} \cdots a'^{m}_{0r^{(m)}}}{a_{00}(-1)^{\frac{1}{2}m(m-1)} D},$$

and in like manner we get

(5) $$\prod_{k=1}^{m} \alpha_k = a^{m-1}_{1r^{(1)}} a^{m-1}_{1r^{(2)}} \cdots a^{m-1}_{1r^{(m)}} / (-1)^{\frac{1}{2}m(m-1)} D.$$

The numerator of the right hand member of this last equality is evidently the resultant (say $R_m$) of $a^{m}_{0x}$ and $a^{m-1}_{1x}$.

Consider next the two differential operators

$$\Delta_1 = m a_{00} \frac{\partial}{\partial a_{10}} + (m-1) a_{01} \frac{\partial}{\partial a_{11}} + \cdots + a_{0m-1} \frac{\partial}{\partial a_{1m-1}},$$

$$\Delta_2 = m a_{0m} \frac{\partial}{\partial a_{1m-1}} + (m-1) a_{0m-1} \frac{\partial}{\partial a_{1m-2}} + \cdots + a_{01} \frac{\partial}{\partial a_{10}};$$

and particularly their effect when applied to $a^{m-1}_{1x}$. We get [see (2)]

(6) $$\Delta_1 a^{m-1}_{1r^{(k)}} = a'^{m}_{0r^{(k)}}, \qquad \Delta_2 a^{m-1}_{1r^{(k)}} = a''^{m}_{0r^{(k)}} = -\frac{r_1^{(k)}}{r_2^{(k)}} a'^{m}_{0r^{(k)}},$$

and from these relations we deduce the following:

(7) $$\Delta_1 \prod_{k=1}^{m} \alpha_k = \frac{\Delta_1 R_m}{(-1)^{\frac{1}{2}m(m-1)} \lfloor 1 D} = a_{00} \Sigma \frac{a^{m-1}_{1r^{(1)}} a^{m-1}_{1r^{(2)}} \cdots a^{m-1}_{1r^{(m-1)}}}{a'^{m}_{0r^{(1)}} a'^{m}_{0r^{(2)}} \cdots a'^{m}_{0r^{(m-1)}}},$$

or, from (1)

(8) $$\frac{\Delta_1 R_m}{(-1)^{\frac{1}{2}m(m-1)} \lfloor 1 D} = \Sigma \alpha_1 \alpha_2 \cdots \alpha_{m-1} r_2^{(m)}.$$

In (7) the symmetric function $\Sigma$ is to be read with reference to the $r$'s, the super-

scripts of the $r$'s replacing the subscripts usual in a symmetric function. Let us now operate with $\Delta_2$ on both members of (7). This gives

$$\frac{\Delta_1\Delta_2 R_m}{(-1)^{\frac{1}{2}m(m-1)}\,\underline{1}\,D} = a_{00}\,\Sigma\,\frac{a^{m-1}_{1r^{(1)}}a^{m-1}_{1r^{(2)}}\cdots a^{m-1}_{1r^{(m-2)}}}{a'^{m}_{0r^{(1)}}a'^{m}_{0r^{(2)}}\cdots a'^{m}_{0r^{(m-2)}}}\left(-\frac{r_1^{(m-1)}}{r_2^{(m-1)}}-\frac{r_1^{(m)}}{r_2^{(m)}}\right).$$

Let $\Sigma_h$ represent an elementary symmetric function of the two groups of homogeneous variables $r_1$, $r_2$ which involves $h$ distinct letters of each group, viz: $r_i^{(m-j+1)}$ $(j=1, 2, \cdots, h)$. Then we have

$$(9)\qquad \frac{\Delta_1\Delta_2 R_m}{(-1)^{\frac{1}{2}m(m-1)}\,\underline{1}\,\underline{1}\,D} = \Sigma[(-1)\alpha_1\alpha_2\cdots\alpha_{m-2}\Sigma_2 r_1^{(m-1)}r_2^{(m)}].$$

We are now in position to prove by induction the following fundamental formula:

$$(10)\qquad \frac{\Delta_1^{m-s-t}\Delta_2^{t}R_m}{(-1)^{\frac{1}{2}m(m-1)}\,\underline{m-s-t}\,\underline{t}\,D} = \Sigma[(-1)^{t}\alpha_1\alpha_2\cdots\alpha_s\Sigma_{m-s}r_1^{(s+1)}r_1^{(s+2)}\cdots r_1^{(s+t)}r_2^{(s+t+1)}\cdots r_2^{(m)}]$$
$$(s=0, 1, \cdots, m;\ t=0, 1, \cdots, m-s),$$

where the outer summation covers all subscripts from 1 to $m$, superscripts of the $r$'s counting as subscripts in the symmetric function. Representing by $J_{m-s-t,t}$ the left hand member of this equality we have from (6)

$$\Delta_2 J_{m-s-t,t} = \Sigma\left((-1)^{t+1}\frac{a^{m-1}_{1r^{(1)}}a^{m-1}_{1r^{(2)}}\cdots a^{m-1}_{1r^{(s-1)}}}{a'^{m}_{0r^{(1)}}a'^{m}_{0r^{(2)}}\cdots a'^{m}_{0r^{(s-1)}}}\right.$$
$$\left.\times\, r_2^{(1)}r_2^{(2)}\cdots r_2^{(s)}\frac{r_1^{(s)}}{r_2^{(s)}}\Sigma_{m-s}r_1^{(s+1)}\cdots r_1^{(s+t)}r_2^{(s+t+1)}\cdots r_2^{(m)}\right).$$

This equals

$$\Sigma(-1)^{t+1}\alpha_1\alpha_2\cdots\alpha_{s-1}S,$$

where $S$ is a symmetric function each term of which involves $t+1$ letters $r_1$ and $m-s-t$ letters $r_2$. The number of terms in an elementary symmetric function of any number of groups of homogeneous variables equals the number of permutations of the letters occurring in any one term when the subscripts (here superscripts) are removed. Hence the number of terms in $\Sigma_{m-s}$ is

$$\frac{\underline{m-s}}{\underline{m-s-t}\,\underline{t}},$$

and the number of terms in $S$ is

$$(m-s+1)\,\underline{m-s}/\underline{t}\,\underline{m-s-t}.$$

But the number of terms in $\Sigma_{m-s+1}(r_1^{(s)} r_1^{(s+1)} \cdots r_1^{(s+t)} r_2^{s-t+1)} \cdots r_2^{m)})$ is

$$\boxed{m-s+1}\,/\,\boxed{m-s-t}\,\boxed{t+1}.$$

Hence

$$S = (t+1)\Sigma_{m-s+1},$$

and so

$$\frac{\Delta_2 J_{m-s-t,\,t}}{t+1} = \Sigma\,[(-1)^{t+1}\alpha_1\alpha_2\cdots\alpha_{s-1}\Sigma_{m-s+1}].$$

This result, with (9), completes the inductive proof of formula (10).

Now the functions $J_{m-s-t,\,t}$ are evidently simultaneous invariants of the binary forms $a_{0x}^m$, $a_{0x}^m$, $a_{0x}^m$, $a_{1x}^{m-1}$. We shall show in the next section that the expressions

$$I_{m-s-t,\,t} \equiv Da_{st} - DJ_{m-s-t,\,t} \quad (s=2,\,3,\,\cdots,\,m\,;\ t=0,\,1,\,\cdots,\,m-s)$$

are, in reality, seminvariants of the form $f_{3m}$ as a whole.

## § 3. *Structure of a ternary form.*

The structure of the right hand member of the fundamental equality (10) shows at once that the general (factorable or non-factorable) quantic $f_{3m}$ $(D \neq 0)$ can be reduced to the following form:

$$(11)\qquad f_{3m} = \prod_{k=1}^{m}(r_2^{(k)}x_1 - r_1^{(k)}x_2 + \alpha_k) + \sum_{s=2}^{m}\sum_{t=0}^{m-s}(a_{st} - J_{m-s-t,\,t})x_1^{m-s-t}x_2^t.$$

This gives explicitly the general "satellite" form of $f_{3m}$, with coefficients expressed rationally in terms of the coefficients of $f_{3m}$. It may be written

$$(12)\qquad D\mu_{m-2} = \sum_{s=2}^{m}\sum_{t=0}^{m-s}\left(Da_{st} - \frac{\Delta_1^{m-s-t}\Delta_2^t R_m}{(-1)^{\frac{1}{2}m(m-1)}\,\boxed{m-s-t}\,\boxed{t}}\right)x_1^{m-s-t}x_2^t = \sum_{s=2}^{m}\sum_{t=0}^{m-s} I_{m-s-t,\,t}x_1^{m-s-t}x_2^t.$$

Now the coefficients $I_{m-s-t,\,t}$ are seminvariants of $f_{3m}$. To fix ideas let $m=3$ and write the usual set of ternary operators,

$$\Omega_{x_1x_2} = a_{01}\frac{\partial}{\partial a_{00}} + 2a_{02}\frac{\partial}{\partial a_{01}} + 3a_{03}\frac{\partial}{\partial a_{02}} + a_{11}\frac{\partial}{\partial a_{10}} + 2a_{12}\frac{\partial}{\partial a_{11}} + a_{21}\frac{\partial}{\partial a_{20}},$$

$$\Omega_{x_2x_1} = 3a_{00}\frac{\partial}{\partial a_{01}} + 2a_{01}\frac{\partial}{\partial a_{02}} + a_{02}\frac{\partial}{\partial a_{03}} + 2a_{10}\frac{\partial}{\partial a_{11}} + a_{11}\frac{\partial}{\partial a_{12}} + a_{20}\frac{\partial}{\partial a_{21}},$$

$$\Omega_{x_3x_1} = a_{20}\frac{\partial}{\partial a_{30}} + 2a_{10}\frac{\partial}{\partial a_{20}} + 3a_{00}\frac{\partial}{\partial a_{10}} + Da_{11}\frac{\partial}{\partial a_{21}} + 2a_{01}\frac{\partial}{\partial a_{11}} + Da_{02}\frac{\partial}{\partial a_{12}},$$

etc. . . . . .

Then $I_{10}$ is annihilated by $\Omega_{x_2x_1}$ but not by $\Omega_{x_1x_2}$, $I_{01}$ is annihilated by $\Omega_{x_1x_2}$ but not by $\Omega_{x_2x_1}$, and $I_{00}$ is annihilated by $\Omega_{x_1x_2}$ but not by $\Omega_{x_2x_1}$. In general $I_{m-s-t,t}$ fails of annihilation when operated upon by a general operator $\Omega_{x_ix_j}$ which contains a partial derivative with respect to $a_{st}$. We have now proved theorem 2.

## § 4. *The semi-discriminants.*

A necessary and sufficient condition that $f_{3m}$ should degenerate into the product of $m$ distinct linear factors is that $\mu_{m-2}$ should vanish identically. Hence, since the number of coefficients in $\mu_{m-2}$ is $\frac{1}{2}m(m-1)$, these equated to zero give a minimum set of conditions in order that $f_{3m}$ should be factorable in the manner stated. As previously indicated we refer to these seminvariants as a set of semi-discriminants of the form $f_{3m}$. They are

$$(13) \qquad I_{m-s-t,t} = Da_{st} - \frac{\Delta_1^{m-s-t}\Delta_2^t R_m}{(-1)^{\frac{1}{2}m(m-1)}\,t\,|\,m-s-t} \qquad \binom{s=2,\,3,\,\cdots,\,m;}{t=0,\,1,\,\cdots,\,m-s}.$$

They are obviously independent since each one contains a coefficient ($a_{st}$) not contained in any other. They are free from adventitious factors, and each one is of degree $2m-1$.

In the case where $m=2$ we have

$$I_{00} = -a_{20}\begin{vmatrix} 2a_{00} & a_{01} \\ a_{01} & 2a_{02} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & 0 \\ 0 & a_{10} & a_{11} \end{vmatrix}.$$

This is also the *ordinary* discriminant of the ternary quadratic.

The three semi-discriminants of the ternary cubic have been computed by the author by another method.* Corresponding results for the case $m=4$ have not been published. They are the following:

$$I_{00} = \tfrac{1}{2^7}a_{40}(4i_1^3 - J_1^2) - R_4,$$

where

$$i_1 = a_{02}^2 - 3a_{01}a_{03} + 12a_{00}a_{04},$$

$$J_1 = 27a_{01}^2a_{04} + 27a_{00}a_{03}^2 + 2a_{02}^3 - 72a_{00}a_{02}a_{04} - 9a_{01}a_{02}a_{03},$$

$$R_4 = \begin{vmatrix} a_{10} & a_{11} & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{10} & a_{11} & a_{12} & a_{13} & 0 \\ 0 & 0 & a_{10} & a_{11} & a_{12} & a_{13} \\ a_{01}a_{10} - a_{00}a_{11} & a_{02}a_{10} - a_{00}a_{12} & a_{03}a_{10} - a_{00}a_{13} & a_{04}a_{10} & 0 & 0 \\ a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & 0 \\ 0 & a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \end{vmatrix},$$

*American Journal of Mathematics, vol. 32 (1910), p. 89.

the other members of the set being obtained by operating upon $R_4$ with powers of $\Delta_1$, $\Delta_2$:

$$\Delta_1 = 4a_{00}\frac{\partial}{\partial a_{10}} + 3a_{01}\frac{\partial}{\partial a_{11}} + 2a_{02}\frac{\partial}{\partial a_{12}} + a_{03}\frac{\partial}{\partial a_{13}},$$

$$\Delta_2 = 4a_{01}\frac{\partial}{\partial a_{13}} + 3a_{03}\frac{\partial}{\partial a_{12}} + 2a_{02}\frac{\partial}{\partial a_{11}} + a_{01}\frac{\partial}{\partial a_{10}},$$

according to the formula

$$I_{4-s-t,\,t} = a_{st}D - \frac{\Delta_1^{4-s-t}\Delta_2^t R_4}{|4-s-t|\,|t|} \qquad (s=2,\,3,\,4\,;\ t=0,\,1,\,\cdots,\,4-s).$$

## § 5. *Proof of Theorem* 1.

The factors of $a_{0x}^m$ being assumed distinct we can always solve $I_{m-s-t,\,t}=0$ for $a_{st}$, the result being obviously rational in the coefficients occurring in $a_{0x}^m$, $a_{1x}^{m-1}$. This proves theorem 1, as far as the case $D \neq 0$ is concerned. Moreover by carrying the resulting values of $a_{st}\,(s=2,\,3,\,\cdots,\,m\,;\ t=0,\,1,\,\cdots,\,m-s)$ back into $f_{3m}$ we get the general form of a ternary quantic which is factorable into linear forms. In the result $a_{0x}^m$, $a_{1x}^{m-1}$ are perfectly general (the former, however, subject to the negative condition $D \neq 0$), whereas

$$(-1)^{\frac{1}{2}m(m-1)}Da_{jx}^{m-j} \equiv \frac{\Delta_1^{m-j}R_m}{|m-j|}x_1^{m-j} + \frac{\Delta_1^{m-j-1}\Delta_2 R_m}{|m-j-1|\,|1|}x_1^{m-j-1}x_2 + \cdots + \frac{\Delta_2^{m-j}R_m}{|m-j|}x_2^{m-j}$$
$$(j=2,\,3,\,\cdots,\,m).$$

Assume next that $D=0$. Then there are two cases to consider. First, $a_{0x}^m$ has multiple factors but $f_{3m}$ as a whole has no multiple linear ternary factors, and a mere interchange of subscripts of the variables $(x_1, x_2, x_3)$ transforms $f_{3m}$ into a new quantic whose binary $a_{0x}^m$ has no multiple factors. For this new quantic, then, $D \neq 0$. Secondly, $f_{3m}$ as a whole may have repeated ternary linear factors. Let there be one such $D$ factor, of multiplicity two. Then $J_{m-s-t,\,t}$, which we now write in the briefer form

$$J_{m-s-t,\,t} = \frac{\Delta_{st}R_m(a_{00},\,a_{01},\,\cdots)}{D(a_{00},\,a_{01},\,\cdots)},$$

is indeterminate. In fact, in this case, two of the $\alpha_i$ in the right hand member of (10), say $\alpha_1$, $\alpha_2$, are to be replaced by one and the same quantity

$$\alpha_{12} \equiv \frac{r_2^{(1)}\dfrac{\partial}{\partial r_1^{(1)}}a_{1r^{(1)}}^{m-1}}{\dfrac{\partial}{\partial r_1^{(1)}}a_{0r^{(1)}}^{\prime m}}.*$$

*See Bulletin of the American Mathematical Society, vol. 17 (1911), p. 451, theorem in § 2.

Then it is not difficult to show that the corresponding true value* of $J_{m-s-t,\,t}$, and hence of $a_{st}$, is

$$a_{st} = \frac{\Delta_{st}\dfrac{\partial}{\partial a_{0i}} R_m(a_{00}, \cdots)}{\dfrac{\partial}{\partial a_{0i}} D(a_{00}, \cdots)} \quad (i = \text{any number of the set } 0, 1, \cdots, m).$$

Likewise, when $f_{3m}$ contains a linear factor of multiplicity three, three of the $\alpha_i$ in (10), say $\alpha_1$, $\alpha_2$, $\alpha_3$ are to be replaced by the same quantity, viz.:

$$\alpha_{123} \equiv \frac{r_2^{(1)}\dfrac{\partial^2}{\partial r_1^{(1)2}} a_{1r^{(1)}}^{m-1}}{\dfrac{\partial^2}{\partial r_1^{(1)2}} a'^{m}_{0r^{1)}}}.$$

and then we get

$$a_{st} = \frac{\Delta_{st}\dfrac{\partial^2}{\partial a_{0i}^2} R_m(a_{00}, \cdots)}{\dfrac{\partial^2}{\partial a_{0i}^2} D(a_{00}, \cdots)} \quad (i = \text{any number of the set } 0, 1, \cdots, m).$$

Similarly in the case of a factor of $f_{3m}$ of multiplicity higher than three. Hence in these cases also $a_{st}$ is rational in the coefficients of $a_{0x}^m$, $a_{1x}^{m-1}$.

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* Cf. SALMON, *Modern Higher Algebra* (fourth edition), p. 90.